From Toda to KdV

D. Bambusi[*], T. Kappeler[†], T. Paul[‡]

May 6, 2015

Abstract

For periodic Toda chains with a large number N of particles we consider states which are N^{-2}-close to the equilibrium and constructed by discretizing arbitrary given C^2-functions with mesh size N^{-1}. Our aim is to describe the spectrum of the Jacobi matrices L_N appearing in the Lax pair formulation of the dynamics of these states as $N \to \infty$. To this end we construct two Hill operators $H_\pm$ – such operators come up in the Lax pair formulation of the Korteweg-de Vries equation – and prove by methods of semiclassical analysis that the asymptotics as $N \to \infty$ of the eigenvalues at the edges of the spectrum of L_N are of the form $\pm(2 - (2N)^{-2}\lambda_n^\pm + \cdots)$ where $(\lambda_n^\pm)_{n\geq 0}$ are the eigenvalues of $H_\pm$. In the bulk of the spectrum, the eigenvalues are $o(N^{-2})$-close to the ones of the equilibrium matrix. As an application we obtain asymptotics of a similar type of the discriminant, associated to L_N.

1 Introduction

It is well known that the (periodic) Toda lattice is an integrable system and by classical *heuristic* arguments, its dynamics are expected to be well described by solutions of the (periodic) KdV equation in the continuous limit (cf [31, 11, 30]). However, only quite recently [29, 4], it has been rigorously proved that in an appropriate asymptotic regime, small solutions of Toda lattices, or more generally,

[*]Dipartimento di Matematica, Università degli Studi di Milano, Via Saldini 50, I-20133 Milano

[†]Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich. Supported in part by the Swiss National Science Foundation.

[‡]CNRS and CMLS, École Polytechnique, F-91128 Palaiseau

of chains of particles with nearest neighbors interaction, referred to as FPU chains, can be described approximately in terms of solutions of the KdV equation. It is important to note that in order to approximate *one* solution of an FPU chain, *two* solutions of the KdV equation are needed, one corresponding to a right moving wave, and the other corresponding to a left moving wave. Furthermore we recall that these results are proved by averaging type methods, allowing to control the dynamical variables for long, but finite intervals of time.

Since the (periodic) Toda lattice is an integrable system, one expects that the above approximation results can be improved in this case by computing the asymptotics of quantities such as the frequencies. In addition, one would like to understand how the integrable structure of Toda lattices is related to the corresponding one of the KdV equation in the continuous limit. In particular, recall that both equations admit a Lax pair formulation. The one for Toda lattices involves Jacobi matrices and the one for the KdV equation Schrödinger operators. In the setup of lattices on the entire line, Toda [30] showed by a formal computation that the continuous limit of Jacobi matrices is given by *one* Schrödinger operator (cf [30], p. 93), leading to *one* solution of the KdV equation. But in view of the rigorous results of [29, 4], *two* solutions of KdV are needed to describe the asymptotics of solutions of Toda lattices in the continuous limit. Hence even on a formal level, Toda's result is incomplete, at least for general initial data. These limitations of Toda's result are also shared by works in the periodic setup such as [17, 18] as well as by studies of other lattices (cf e.g. [25]). Indeed, in [17], p. 587, the author points out that he only considers very special initial data of the periodic Toda lattice.

The formal results of Toda et al. and the rigorous results of [29, 4] lead to the problem of how to construct two Schrödinger operators yielding the two KdV solutions needed to describe the asymptotics of Toda lattices in the continuous limit without the restrictions on the initial data mentioned above. In the present paper, we solve this problem in the periodic setup in which case these operators are also referred to as Hill operators. It might come as a surprise that they are constructed by some methods of semiclassical analysis. One of the main results we show says that they can be used to approximately describe the limiting asymptotics of the spectra of periodic Jacobi matrices.

We believe that our results and the methods developed for proving them will be an essential tool for studying all kinds of properties of the asymptotics of Toda lattices in the continuous limit. Results in this direction are obtained in [3] by applying what is proved in the present paper: one of the results of [3] provides the first two terms in the asympotics of the Toda frequencies in terms of the KdV frequencies corresponding to the *two* Hill operators, mentioned above.

Finally we would like to discuss the connection of the present research with the so called FPU problem, which actually is the main motivation for our research.

We recall that in their celebrated report [13], Fermi, Pasta, and Ulam studied the dynamics of FPU chains. They wanted to confirm numerically that energy sharing among the different degrees of freedom occurs. Very much to their surprise, they observed recurrent dynamics instead. It led to the question, referred to as the FPU problem, whether the observed recurrence phenomenon persists in the thermodynamic limit. In case it does, it would contradict the so called equipartition principle leading to potentially serious problems for the foundations of classical statistical mechanics. We emphasize that, notwithstanding the huge number of computations and the enormous amount of theoretical work done up till now (see e.g. [6], [8], [24], [27]), an answer to this problem is still not known. For status reports on the research of FPU chains see [7], [10], [16].
In the context of the FPU problem, our interest in Toda lattices stems from the facts that on the one side, Toda lattices are integrable and hence their continuous limits might be easier to study and at a deeper level, due to the additional structures present, and that on the other side, near the equilibrium, FPU chains are well approximated by Toda chains. Indeed, it was already pointed out in [12] that close to the equilibrium, (periodic) FPU chains are typically better approximated by (periodic) Toda chains than by the linear model. Subsequently, numerical evidence was found that up to a long time, periodic solutions of FPU chains with small initial data are very well approximated by Toda chains. See the quite recent work in this direction [5] and references therein.

2 Statement of main results

The Toda lattice, in the setting of periodic boundary conditions with period $N \geq 2$, is the Hamiltonian system with Hamiltonian

$$\mathcal{H} = \frac{1}{2}\sum_{n=1}^{N} p_n^2 + \sum_{n=1}^{N} e^{q_n - q_{n+1}}.$$

Here q_n denotes the displacement from the equilibrium position of the n'th particle, p_n its momentum and (q_n, p_n) is defined for any n in $\mathbb{Z}$ by requiring that $(q_{i+N}, p_{i+N}) = (q_i, p_i)$ for any $i \in \mathbb{Z}$. When expressed in Flaschka coordinates, $b_n = -p_n$ and $a_n = e^{\frac{1}{2}(q_n - q_{n+1})}$([14]), the Hamiltonian equations of motion associated to $\mathcal{H}$ take a Lax pair formalism description given by

$$\dot{L} = [B, L] \tag{2.1}$$

where the $N \times N$ matrices $L = L(b,a)$ and $B = B(a)$ are of the form

$$\begin{pmatrix} b_1 & a_1 & 0 & \dots & 0 & a_N \\ a_1 & b_2 & a_2 & \dots & 0 & 0 \\ 0 & a_2 & b_3 & \dots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & \dots & & \dots & b_{N-1} & a_{N-1} \\ a_N & 0 & \vdots & & a_{N-1} & b_N \end{pmatrix} \text{ and } \begin{pmatrix} 0 & a_1 & 0 & \dots & 0 & -a_N \\ -a_1 & 0 & a_2 & \dots & 0 & 0 \\ 0 & -a_2 & 0 & \dots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & \dots & & \dots & 0 & a_{N-1} \\ a_N & 0 & \vdots & & -a_{N-1} & 0 \end{pmatrix}$$

respectively, with $a = (a_n)_{1 \leq n \leq N} \in \mathbb{R}^N_{>0}$ and $b = (b_n)_{1 \leq n \leq N} \in \mathbb{R}^N$. Notice that the matrix $L(0_N, 1_N)$ with $b = 0_N = (0, ..., 0)$ and $a = 1_N = (1, ..., 1)$ is an equilibrium for (2.1). We are interested in the $N \to \infty$ asymptotics of various spectral quantities of $L(b^N, a^N)$ for b^N, a^N of the form

$$b_n^N = \varepsilon\beta\big(\frac{n}{N}\big) \quad \text{and} \quad a_n^N = 1 + \varepsilon\alpha\big(\frac{n}{N}\big) \tag{2.2}$$

where ε is a *coupling parameter* and α, β are functions in $C_0^2(\mathbb{T}, \mathbb{R})$, i.e. $1-$periodic C^2-functions with $[\alpha] = [\beta] = 0$ with $[\alpha]$ denoting the mean of α, $[\alpha] = \int_0^1 \alpha(x)dx$. Alternatively, one can consider

$$p_n^N = -\varepsilon\beta\big(\frac{n}{N}\big) \quad \text{and} \quad q_n^N = -2N\varepsilon\xi\big(\frac{n}{N}\big)$$

where ξ is the element in $C_0^3(\mathbb{T})$, satisfying $\xi' = \alpha$. Using that

$$\exp\Big(\frac{q_n^N - q_{n+1}^N}{2}\Big) = 1 + \frac{q_n^N - q_{n+1}^N}{2} + O(\epsilon/N) = a_n^N + O(\epsilon/N)$$

one can show that our results stated below hold for either of the two discretizations. The limiting equations strongly depend on the choice of the coupling parameter ϵ. In [9] it is shown that with $\epsilon \sim 1$, one obtains in the limit as $N \to \infty$ a nonlinear system of equations of hyperbolic type, which contains as a special case the inviscid Burgers equation. In contrast to [9], we choose $\varepsilon \equiv \varepsilon_N = (2N)^{-2}$. It turns out that in this case, the asymptotics of the dynamics is described in terms of two solutions of the KdV equation (cf [3]). Our aim is to compute the asymptotics of the eigenvalues of $L(b^N, a^N)$ and of the corresponding discriminant as $N \to \infty$. Let us note that in view of the Lax pair representation, the spectrum of $L(b^N, a^N)$ is conserved by the Toda flow. To obtain a set of independent integrals of motion it turns out to be more convenient (see e.g. [19]) to double the size of $L(b^N, a^N)$ and to consider

$$Q_N^{\alpha,\beta} \equiv Q(b^N, a^N) = L\big((b^N, b^N), (a^N, a^N)\big), \tag{2.3}$$

namely

$$Q_N^{\alpha,\beta} = \begin{pmatrix} b_1^N & a_1^N & 0 & \dots & \dots & \dots & \dots & \dots & 0 & a_N^N \\ a_1^N & b_2^N & a_2^N & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & a_2^N & b_3^N & a_3^N & 0 & \dots & \dots & \dots & \dots & 0 \\ \vdots & & & & & & & & & \vdots \\ 0 & \dots & 0 & a_{N-1}^N & b_N^N & a_N^N & 0 & \dots & \dots & 0 \\ 0 & \dots & \dots & 0 & a_N^N & b_1^N & a_1^N & 0 & \dots & 0 \\ \vdots & & & & & & & & & \vdots \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 & a_{N-2}^N & b_{N-1}^N & a_{N-1}^N \\ a_N^N & 0 & \dots & \dots & \dots & \dots & \dots & 0 & a_{N-1}^N & b_N^N \end{pmatrix}$$

The eigenvalues of $Q_N^{\alpha,\beta}$ when listed in increasing order and with multiplicities satisfy

$$\lambda_0^N < \lambda_1^N \leq \lambda_2^N < \cdots < \lambda_{2N-3}^N \leq \lambda_{2N-2}^N < \lambda_{2N-1}^N.$$

By Floquet theory (cf. e.g. [19]) one sees that in the case where N is even, $\lambda_0, \lambda_3, \lambda_4, \dots, \lambda_{2N-5}, \lambda_{2N-4}, \lambda_{2N-1}$ are the N eigenvalues of $L(b^N, a^N)$. For N odd, they are given by $\lambda_1, \lambda_2, \lambda_5, \lambda_6, \dots, \lambda_{2N-5}, \lambda_{2N-4}, \lambda_{2N-1}$. To describe the asymptotics of λ_n^N at the edges, $n \sim 1$ or $n \sim 2N-1$, we need to introduce two Hill operators $H_\pm := -\partial_x^2 + q_\pm$ with potentials

$$q_\pm(x) = -2\alpha(2x) \mp \beta(2x). \tag{2.4}$$

The discovery of these two operators and of their role in the description of the asymptotics as $N \to \infty$ of the spectrum of $Q_N^{\alpha,\beta}$ is one of the main contributions of this paper. The role played by the operators H_- and H_+ in the description of the asymptotics of the left respectively right edge of the spcetrum of $Q_N^{\alpha,\beta}$ will be explained in detail in Section 7. Furthermore we point out that in our subsequent paper [3] we prove that the limiting dynamics of the Toda lattices (b^N, a^N) as $N \to \infty$ can be described in terms of the solutions of the KdV equation corresponding to q_- and q_+. Note that the two potentials q_- and q_+ determine α and β uniquely and that they are independent from each other. Loosely speaking, in terms of the asymptotics described in [3], it means that the parts of the Toda lattices corresponding to the spectrum at the two edges do not interact although Toda lattices are nonlinear systems. The formulas (2.4) for q_- and q_+ are an outcome of semiclassical analysis, discussed in Section 4 – see also the explanations below after Remark 2.2 .
Note that $q_\pm$ are periodic functions of period $1/2$. The periodic eigenvalues $(\lambda_n^\pm)_{n\geq 0}$ of $H_\pm$ on $[0,1]$, when listed in increasing order and with multiplicities, are known to satisfy $\lambda_0^\pm < \lambda_1^\pm \leq \lambda_2^\pm < \cdots$. It turns out that the asymptotics of

the eigenvalues of $Q_N^{\alpha,\beta}$ exhibit three different regions: the bulk and the two edges, which shrink to $\{-2\}$ and $\{+2\}$, respectively, as $N \to \infty$. Each of these three parts of the spectrum has its proper asymptotics: in the bulk, the spectrum is close to the one of the equilibrium matrix by a distance smaller than the distance between the given Jacobi matrices and the equilibrium matrix, whereas in each of the two edges, the first correction is of the same order as this distance and involves the spectrum of one of the two Hill operators $H_\pm$.
To define the two edges of the spectrum consider a function $F : \mathbb{N} \to \mathbb{R}_{\geq 1}$ satisfying

$$(F) \qquad \lim_{N\to\infty} F(N) = \infty; \qquad \text{increasing;} \qquad F(N) \leq N^\eta \text{ with } 0 < \eta < 1/2.$$

Theorem 2.1 *Let F satisfy* (F) *and let $\alpha, \beta \in C_0^2(\mathbb{T}, \mathbb{R})$. Then the asymptotics of λ_n^N are as follows:*
at the left edge*: for $0 \leq n \leq 2[F(N)]$*

$$\lambda_n^N = -2 + \frac{1}{4N^2}\lambda_n^- + O(F(N)^2 N^{-3})$$

at the right edge*: for $0 \leq n \leq 2[F(N)]$*

$$\lambda_{2N-1-n}^N = 2 - \frac{1}{4N^2}\lambda_n^+ + O(F(N)^2 N^{-3})$$

in the bulk*: for $n = 2\ell, 2\ell - 1$ with $[F(N)] < \ell < N - [F(N)]$,*

$$\lambda_n^N = -2\cos\frac{\ell\pi}{N} + O(N^{-2}F(N)^{-1}).$$

These estimates hold uniformly in $0 \leq n \leq 2N-1$ and uniformly on bounded subsets of functions α, β in $C_0^2(\mathbb{T}, \mathbb{R})$.

Remark 2.2 *In the case where $F(N) = N^\eta, 0 < \eta < 1/2$, the asymptotics of Theorem 2.1 read as follows:*

$$\lambda_n^N = -2 + \frac{1}{4N^2}\lambda_n^- + O(N^{-3+2\eta}), \quad \forall\, 0 \leq n \leq 2[N^\eta]$$

$$\lambda_{2N-1-n}^N = 2 - \frac{1}{4N^2}\lambda_n^+ + O(N^{-3+2\eta}), \quad \forall\, 0 \leq n \leq 2[N^\eta]$$

$$\lambda_{2\ell}^N, \lambda_{2\ell-1}^N = -2\cos\frac{\ell\pi}{N} + O(N^{-2-\eta}), \quad \forall\, [N^\eta] < n < N - [N^\eta].$$

To prove Theorem 2.1 we use singular perturbation methods, more specifically methods from semiclassical approximation. Indeed it has been proved in [9] that Jacobi matrices such as $Q_N^{\alpha,\beta}$ can be viewed as matrix representations of certain semiclassical Toeplitz operators T_N in the framework of the geometric quantization of the 2d torus. Note that the Jacobi matrices $Q_N^{\alpha,\beta}$ – and hence the associated Toeplitz operators – are perturbations of size $\frac{1}{N^2}$ of the equilibrium matrix $Q(0_N, 1_N)$ whose spectrum is $\{-2\cos\frac{l\pi}{N},\ l = 0, \dots, N\}$. Since $\cos\frac{l\pi}{N} - \cos\frac{(l-1)\pi}{N} = O(N^{-2})$ for $l \sim 1$ or $l \sim N$, the size of the perturbation is of the same order as the spacing between the unperturbed eigenvalues so that regular perturbation methods fail. Using semiclassical methods, we compute the asymptotics of the eigenvalues at the two edges of the spectrum by constructing semiclassical (Lagrangian) quasimodes for T_N, for which the two Hill operators appear in the transport equation associated to the construction. As customary for the quantization of compact symplectic manifolds, the Toeplitz operators T_N act on a Hilbert space of dimension $2N$ with N playing the role of t he inverse of an effective Planck constant. In the bulk, i.e. for $1 \ll l \ll N$, one has $|\cos\frac{l\pi}{N} - \cos\frac{(l-1)\pi}{N}| \gg N^{-2}$ and thus regular perturbation methods apply. Finally let us mention that our method allows to obtain the full asymptotic expansion in $\frac{1}{N^2}$ of the entire spectrum – see the discussion at the end of Section 8.

As an application of Theorem 2.1 we derive asymptotics for the characteristic polynomial $\chi_N(\mu)$ of $Q_N^{\alpha,\beta}$ as $N \to \infty$. Note that $\chi_N(\mu)$ gives rise to the spectral curve $\{(\mu, z) \in \mathbb{C}^2 | z^2 = \chi_N(\mu)\}$ which plays an important role in the theory of periodic Toda lattices. These asymptotics will be of great use in the subsequent work [3]. By Floquet theory, $\chi_N(\mu)$ can be expressed in terms of the discriminant associated to the difference equation ($k \in \mathbb{Z}$)

$$a_{k-1}^N y(k-1) + b_k^N y(k) + a_k^N y(k+1) = \mu y(k). \tag{2.5}$$

Indeed denote by y_1^N and y_2^N the fundamental solutions of (2.5) determined by

$$y_1^N(0,\mu) = 1,\ y_1^N(1,\mu) = 0 \quad \text{and} \quad y_2^N(0,\mu) = 0,\ y_2^N(1,\mu) = 1.$$

The discriminant of (2.5) is then defined as the trace of the Floquet matrix associated to (2.5) and given by

$$\Delta_N(\mu) = y_1^N(N,\mu) + y_2^N(N+1,\mu).$$

In view of the Wronskian identity, μ is an eigenvalue of $L(b^N, a^N)$ $[Q_N^{\alpha,\beta}]$ iff $\Delta_N(\mu) - 2 = 0$ $[\Delta_N^2(\mu) - 4 = 0\}]$. Hence up to a multiplicative constant, $\Delta_N^2 - 4$ and χ_N coincide. From the recursive formula for $y_2^N(k,\mu)$ one then sees ([19]) that $\Delta_N^2(\mu) - 4 = q_N^{-2}\chi_N(\mu)$ where $q_N = \prod_1^N (1 + \frac{1}{4N^2}\alpha(\frac{n}{N}))$.

Analogously, denote by $\Delta_{\pm}(\lambda) \equiv \Delta(\lambda, q_{\pm})$ the discriminant of

$$-y''(x,\lambda) + q_{\pm}(x)y(x,\lambda) = \lambda y(x,\lambda) \tag{2.6}$$

defined as the trace of the Floquet operator associated to (2.6),

$$\Delta_{\pm}(\lambda) = y_1^{\pm}(1/2,\lambda) + (y_2^{\pm})'(1/2,\lambda)$$

where $y_1^{\pm}(x,\lambda)$ and $(y_2^{\pm})'(x,\lambda)$ are the fundamental solutions of (2.6) defined by

$$y_1^{\pm}(0,\lambda) = 1,\ (y_1^{\pm}(0,\lambda))' = 0 \quad \text{and} \quad y_2^{\pm}(0,\lambda) = 0,\ (y_2^{\pm}(0,\lambda))' = 1.$$

Similarly as in the case of the Toda lattice, λ is a periodic eigenvalue of $H_{\pm}$ on the interval $[0,1]$ iff $\Delta_{\pm}^2((\lambda)-4 = 0$. Note that $\Delta_{\pm}^2((\lambda)-4$ is an entire function and can be viewed as a regularized determinant of $H_{\pm}$, referred to as characteristic function of (2.6). It leads to the spectral curves $\{(\lambda,z) \in \mathbb{C}^2 |\, z^2 = \Delta_{\pm}^2(\lambda) - 4\}$ which play an important role in the theory of the KdV equation. We will state our result on the asymptotics of χ_N in terms of the discriminant Δ_N. With $M = [F(N)]$ and F as before, let $\Lambda^{\pm,M} \equiv \Lambda_2^{\pm,M}$ be the box

$$\Lambda^{\pm,M} := [\lambda_0^{\pm} - 2, \lambda_{2[F(M)]}^{\pm} + 2] + i[-2,2]$$

and choose $N_0 \in \mathbb{Z}_{\geq 1}$ so that

$$\lambda_{2k+1}^{\pm} - \lambda_{2k}^{\pm} \geq 6 \quad \forall\, k \geq F(F(N_0)) \tag{2.7}$$

By the Counting Lemma for periodic eigenvalues (cf e.g. [22]), N_0 can be chosen uniformly for bounded subsets of function α, β in $C_0^2(\mathbb{T},\mathbb{R})$.

Theorem 2.3 *Let F satisfy* (F), *$M = [F(N)]$ with $N \geq N_0$, and $\alpha, \beta \in C_0^2(\mathbb{T},\mathbb{R})$. Then uniformly for λ in $\Lambda^{-,M}$*

$$\Delta_N\big(-2 + \frac{1}{4N^2}\lambda\big) = (-1)^N \Delta_-(\lambda) + O\left(\frac{F(M)^2}{M}\right). \tag{2.8}$$

Similarly, uniformly for λ in $\Lambda^{+,M}$

$$\Delta_N(2 - \frac{1}{4N^2}\lambda) = \Delta_+(\lambda) + O\left(\frac{F(M)^2}{M}\right). \tag{2.9}$$

These estimates hold uniformly on bounded subsets of functions α and β in $C_0^2(\mathbb{T},\mathbb{R})$.

Remark 2.4 *In the case where $F(N) = N^\eta, 0 < \eta < 1/2$, the asymptotics of Theorem 2.3 read as follows:*

$$\Delta_N\big(-2+\frac{1}{4N^2}\lambda\big) = (-1)^N\Delta_-(\lambda) + O\left(N^{-\eta(1-\eta)}\right), \quad \forall\,\lambda \in \Lambda^{-,N^\eta},$$

$$\Delta_N(2-\frac{1}{4N^2}\lambda) = \Delta_+(\lambda) + O\left(N^{-\eta(1-\eta)}\right), \quad \forall\,\lambda \in \Lambda^{+,N^\eta}.$$

We remark that we did not aim at getting the maximal range of the λ's for which (2.8) and (2.9) hold. Moreover, although we didn't state such a result here, our method allows to compute the asymptotics of the discriminant for λ in the bulk region as well. In the companion paper [3], the results of Theorems 2.1 and 2.3 are used as an important ingredient for computing the asymptotics of frequencies and actions of Toda lattices in terms of the corresponding quantities of the KdV equation.

Organisation of paper: The proof of Theorem 2.1 relies on the construction of quasimodes for the Jacobi matrices $Q_N^{\alpha,\beta}$. This construction is done in the framework of the geometric quantization of the torus (Section 3). The matrices $Q_N^{\alpha,\beta}$ are shown to be the matrix representation of Toeplitz operators (Section 4), whose action on a certain type of Lagrangian states is described in detail in Theorem 4.4. Proposition 5.1 in Section 5 states an abstract result on the construction of quasimodes that we couldn't find in the literature and which is crucial for the proof of Theorem 2.1. The two cases corresponding to the bulk and the edges of the spectrum are treated in Section 6 and Section 7 respectively. The proof of Theorem 2.1 is summarized in Section 8. In Section 9 we first compute the asymptotics of the Casimir functionals of the Toda lattice (Proposition 9.1) and then, using Theorem 2.1, obtain the asymptotics of the discriminant of $Q_N^{\alpha,\beta}$ in terms of the discriminants of $H_\pm$, stated in Theorem 2.3. In addition, we apply Theorem 2.3 to prove similar asymptotics for the derivatives of $\Delta_N(\mu)$ and to derive aymptotics of the zeroes of $\partial_\mu\Delta_N(\mu)$ at the two edges.

Methods: The methodology used in this paper, based on the geometric quantization of the torus, is strongly inspired by [9]. In that paper, the authors consider the large N asymptotics of Toda lattices, both for Dirichlet and periodic boundary conditions, in the case where the a_n's and b_n's are given by the discretization of regular functions, i.e. the coupling parameter ϵ equals 1, and they derive the limiting PDE.

Finally we mention that this work has been announced in [2].

Acknowledgments: The authors would like to thank the University of Milan, the Swiss National Science Foundation, the University of Zürich, the CNRS and the École polytechnique for financial support during the elaboration of this work.

3 Geometric quantization of $\mathbb{T}^2$

The geometric quantization of the two dimensional torus (resp. sphere) and the underlying so-called Toeplitz operators theory has been shown in [9] to be a natural set-up for studying the large N limit of the Toda lattice with periodic (resp. Dirichlet) boundary conditions. Although most of the computations in the present papers are going to be carried out from scratch, we recall in this section the basic facts concerning Toeplitz operators.

Consider the standard 2-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, identified with $\mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$, with canonical symplectic form $\omega = dx \wedge dy$ and Planck constant $(4\pi N)^{-1}$. Let $E \to \mathbb{T}^2$ be a holomorphic line bundle with connection $\nabla = d - 2\pi i x dy$ and denote by κ the curvature form, $\kappa = d(-2\pi i x dy)$. Then $\frac{i}{2\pi}\kappa = \omega$. In particular, the Chern class of E, given by the cohomology class $\left[\frac{i}{2\pi}\kappa\right]$, satisfies $\left[\frac{i}{2\pi}\kappa\right] = [\omega]$. Denote by $(\mathcal{H}_{2N}^{\sim}, \langle\cdot,\cdot\rangle_{\sim})$ the Hilbert space whose elements are holomorphic sections $\mathbb{T}^2 \to E^{\otimes 2N}$, viewed as entire functions $f : \mathbb{C} \to \mathbb{C}$ satisfying

$$f(z+m+in) = e^{2N\pi[z(m-in)+\frac{1}{2}(m^2+n^2)]} f(z) \qquad \forall (m,n) \in \mathbb{Z}^2,\ z \in \mathbb{C}.$$

The inner product is given by $\langle f,g\rangle_{\sim} = \int_{[0,1]^2} f(z)\overline{g(z)} e^{-2N\pi|z|^2} dxdy$. We identify $\mathcal{H}_{2N}^{\sim}$ (isometrically) with the space $\mathcal{H}_{2N}$ of theta functions of order $2N$, i.e. entire functions $f : \mathbb{C} \to \mathbb{C}$, satisfying

$$f(z+m+in) = e^{2N\pi(n^2-2inz)} f(z) \qquad \forall (m,n) \in \mathbb{Z}^2,\ z \in \mathbb{C}$$

with inner product $\langle f,g\rangle = \int_{[0,1]^2} f(z)\overline{g(z)} e^{-4\pi N y^2} dxdy$. For $0 \le j \le 2N-1$, let

$$\theta_j^N(z) = (4N)^{1/4} e^{-\pi j^2/2N} \sum_{n\in\mathbb{Z}} e^{-\pi(2Nn^2+2jn)} e^{2\pi i z(j+2Nn)}. \tag{3.1}$$

One verifies that $(\theta_j^N)_{0\le j\le 2N-1}$ is an orthonormal basis of $\mathcal{H}_{2N}$. Observe that in contrast to the 'standard' case of the quantization of a cotangent bundle, the Hilbert space $\mathcal{H}_{2N}$ is finite dimensional. From a point of view of physics, this fact is justified by the Heisenberg uncertainty principle. The Toeplitz quantization of a function $F : \mathbb{T}^2 \to \mathbb{R}$ is given by the sequence of operators $T_F^N : \mathcal{H}_{2N} \to \mathcal{H}_{2N},\ f \mapsto P_N(fF)$ where $P_N : (L^2([0,1]^2, e^{-4N\pi y^2} dxdy) \to \mathcal{H}_{2N}$ denotes the orthogonal projector,

$$(P_N f)(z) = \sum_{j=0}^{2N-1} \langle f, \theta_j^N\rangle \theta_j^N(z).$$

More generally, a Toeplitz operator is a sequence of operators $(T^N)_{N\ge 1}$ where for $N \ge 1, T^N : \mathcal{H}_{2N} \to \mathcal{H}_{2N}$ is an operator of the form $T^N \sim \sum_{j=0}^{\infty} N^{-j} T_{S_j}^N$. The function $S_0 : \mathbb{T}^2 \to \mathbb{R}$ is referred to as principal symbol.

4 Jacobi matrices as Toeplitz operators

In this section we study Jacobi matrices with entries defined in terms of discretizations of functions of a certain regularity, from a 'Toeplitz operator' perspective. In particular we show how their action on certain families of elements in the Hilbert spaces $\mathcal{H}_{2N}$, referred to as Lagrangian states, can be explicitly described in terms of differential operators – see Theorem 4.4 below. This theorem is key for our construction of quasimodes.
For α, β in $C_0^2(\mathbb{T}, \mathbb{R})$ and $N \geq 3$, denote by $T_N^{\alpha,\beta}$ the linear operator on $\mathcal{H}_{2N}$ whose representation with respect to the basis $[\theta_{2N-1}^N, \ldots, \theta_0^N]$ is $Q_N^{\alpha,\beta}$. As an example, consider the operator $T_N^{0,0}$. To study its properties let us begin by recording the following elementary result.

Lemma 4.1 *$\big((\mp 1)^n\big)_{0 \leq n \leq 2N-1}$ is an eigenvector of $Q_N^{0,0}$ corresponding to the eigenvalue ∓ 2, and, for any $1 \leq \ell \leq N-1$, the vectors $\big(e^{i\pi(N-\ell)n/N}\big)_{0 \leq n \leq 2N-1}$ and $\big(e^{i\pi(N+\ell)n/N}\big)_{0 \leq n \leq 2N-1}$ are eigenvectors of $Q_N^{0,0}$ corresponding to the eigenvalue $-2\cos\frac{\ell\pi}{N}$. They form an orthogonal basis in $\mathbb{C}^{2N}$.*

From Lemma 4.1 it follows that $\psi^{N,k}(z),\ 0 \leq k \leq 2N-1$, is an orthonormal basis of $\mathcal{H}_{2N}$ of eigenfunctions of $T_N^{0,0}$, $T_N^{0,0}\psi^{N,k} = 2\cos\frac{k\pi}{N}\psi^{N,k}$, where

$$\psi^{N,k}(z) = (2N)^{-1/2}\sum_{n=1}^{2N} e^{i\pi\frac{kn}{N}}\theta_{2N-n}^N(z) = (2N)^{-1/2}\sum_{n=0}^{2N-1} e^{-i\pi\frac{kn}{N}}\theta_n^N(z). \tag{4.1}$$

Alternatively, $\psi^{N,k}$ can be expressed with the help of the kernel

$$\rho_N(z,w) = \sum_{j=0}^{2N-1}\theta_j^N(z)\overline{\theta_j^N(w)}. \tag{4.2}$$

Lemma 4.2 *For any $0 \leq k \leq 2N-1$,*

$$\psi^{N,k}(z) = (4N)^{-1/4}\int_0^1 \rho_N\big(z, k/2N + is\big)e^{-2N\pi s^2}\,ds.$$

Proof: In view of (3.1) and (4.2), the claimed identity follows easily from the identity $\sum_{n\in\mathbb{Z}}\int_0^1 e^{-2\pi N(n+s+\frac{j}{2N})^2}ds = \int_{-\infty}^{\infty} e^{-(\sqrt{2\pi N}x)^2}dx = (2N)^{-1/2}$. □

It is useful to introduce for an arbitrary real or complex valued function $f \in L^2(\mathbb{T})$ and $0 \leq k \leq 2N-1$,

$$\psi_f^{N,k}(z) = (4N)^{-1/4}\int_0^1 f(s)\rho_N\big(z, k/2N + is\big)e^{-2N\pi s^2}\,ds. \tag{4.3}$$

It is an element in $\mathcal{H}_{2N}$. For $f \in L^2(\mathbb{T})$, denote by $\hat{f}_n$ the n'th Fourier coefficient of f, $\hat{f}_n = \int_0^1 f(x) e^{-i2\pi n x} dx$ and by $\|f\|_\ell$ the ℓ-th Sobolev norm $\|f\|_\ell = \Big(\sum_{n\in\mathbb{Z}} |\hat{f}_n|^2 (1+|n|)^{2\ell}\Big)^{1/2}$. Further denote by $\|f\|_{C^\ell}$ the following norm of $f \in C^\ell(\mathbb{T})$: $\|f\|_{C^\ell} = \sup\limits_{0\le x\le 1} \sum_{j=0}^{\ell} |\partial_x^j f(x)|$.

Lemma 4.3 *For $f, g \in L^2(\mathbb{T})$ and $0 \le k, \ell \le 2N-1$*

(i) $\psi_f^{N,k}(z) = \frac{1}{\sqrt{2N}} \sum_{j=0}^{2N-1} \theta_j^N(z) e^{-i\pi kj/N} \sum_{m\in\mathbb{Z}} \hat{f}_m e^{-\pi m^2/2N} e^{-i\pi mj/N}$.
Alternatively, with $\Delta = -d^2/dx^2$,

$$\psi_f^{N,k}(z) = \frac{1}{\sqrt{2N}} \sum_{j=0}^{2N-1} \big(e^{-\Delta/8\pi N} f\big)\big(-j/2N\big) e^{-i\pi kj/N} \theta_j^N(z). \tag{4.4}$$

(ii) $\langle \psi_f^{N,k}, \psi_g^{N,\ell} \rangle = \sum_{n\in\mathbb{Z}} \hat{f}_n \overline{\hat{g}_{n+k-\ell}} e^{-\pi n^2/2N} e^{-\pi(n+k-\ell)^2/2N}$.

(iii) $\big| \langle \psi_f^{N,k}, \psi_g^{N,k} \rangle - \langle f, g \rangle \big| \le \frac{1}{4\pi N} \|f'\|_0 \cdot \|g'\|_0 \quad \forall\, f, g \in H^1(\mathbb{T})$.

(iv) *The linear maps $L^2(\mathbb{T}) \to \mathcal{H}_{2N}, f \mapsto \psi_f^{N,k}$, are bounded, $\|\psi_f^{N,k}\| \le \|f\|_0$.*

Proof: (i) By the definitions of $\psi_f^{N,k}$ and ρ_N

$$\psi_f^{N,k}(z) = (4N)^{-1/4} \sum_{j=0}^{2N-1} \theta_j^N(z) \int_0^1 f(s)\, \overline{\theta_j^N(k/2N + is)}\, e^{-2N\pi s^2} ds.$$

Using the definition (3.1) of θ_j^N one gets

$$(4N)^{-1/4} \int_0^1 f(s) \overline{\theta_j^N(\frac{k}{2N} + is)} e^{-2N\pi s^2} dx = e^{-i\pi \frac{kj}{N}} \sum_{n\in\mathbb{Z}} \int_0^1 e^{-2\pi N(n+s+\frac{j}{2N})^2} f(s) ds. \tag{4.5}$$

As $\sum_{n\in\mathbb{Z}} e^{-2\pi N(n+s+j/2N)^2} = \frac{1}{\sqrt{2N}} \sum_{n\in\mathbb{Z}} e^{2\pi n(s+j/2N)} e^{-\pi n^2/2N}$ (Poisson summation formula) it then follows that

$$(4N)^{-1/4} \int_0^1 f(s) \overline{\theta_j^N\big(k/2N + is\big)} e^{-2N\pi s^2} ds = \frac{1}{\sqrt{2N}} \sum_{n\in\mathbb{Z}} e^{i\pi\frac{(n-k)j}{N}} e^{-\pi \frac{n^2}{2N}} \hat{f}_{-n},$$

yielding the claimed formula. To verify the alternative formula, note that by (4.5) and the fact that f is periodic, one has

$$\begin{aligned}\psi_f^{N,k}(z) &= \sum_{j=0}^{2N-1} \theta_j^N(z) e^{-i\pi kj/N} \sum_{n\in\mathbb{Z}} \int_0^1 e^{-2\pi N(n+s+j/2N)^2} f(s)ds \\ &= \sum_{j=0}^{2N-1} \theta_j^N(z) e^{-i\pi kj/N} \int_{-\infty}^{\infty} e^{-2\pi N(y+j/2N)^2} f(y)dy.\end{aligned}$$

Evaluating the heat flow for the initial data f at $x = -\frac{j}{2N}$ and time $t = \frac{1}{4}\frac{1}{2\pi N}$ we obtain the claimed identity (4.4).

(ii) By the definition of ρ_N and the fact that $(\theta_j^N)_{0\le j\le 2N-1}$ is an orthonormal basis of $\mathcal{H}_{2N}$, we have that

$$\begin{aligned}&\int_0^1 \rho_N\big(x+iy, k/2N+is\big)\rho_N\big(x+iy, \ell/2N+it\big)e^{-4\pi Ny^2}dxdy \\ &= \sum_{j,n} \overline{\theta_j^N\big(k/2N+is\big)}\theta_n^N\big(\ell/2N+it\big)\langle\theta_j^N, \theta_n^N\rangle = \sum_{j=0}^{2N-1} \overline{\theta_j^N\big(\frac{k}{2N}+is\big)}\theta_n^N\big(\frac{\ell}{2N}+it\big).\end{aligned}$$

Hence $\langle \psi_f^{N,k}, \psi_g^{N,\ell}\rangle$ is equal to

$$\sum_{j=0}^{2N-1} \frac{1}{\sqrt{4N}} \int_0^1 \overline{\theta_j^N\big(k/2N+is\big)} f(s) e^{-2N\pi s^2} ds \int_0^1 \theta_j^N\big(k/2N+it\big)\overline{g(t)} e^{-2N\pi t^2} dt.$$

By (i) and the fact that $\sum_{j=0}^{2N-1} e^{i\pi(\ell-n-k+m)j/N} = 2N\delta_{\ell-n,k-m}$ we get that

$$\langle \psi_f^{N,k}, \psi_g^{N,\ell}\rangle = \sum_{m\in\mathbb{Z}} \hat{f}_{-m}\overline{\hat{g}_{-m+k-\ell}} e^{-\pi m^2/2N} e^{-\pi(-m+k-\ell)^2/2N}.$$

(iii) From item (ii) it follows that for $k = \ell$,

$$\langle \psi_f^{N,k}, \psi_g^{N,k}\rangle = \sum_{n\in\mathbb{Z}} \hat{f}_n\overline{\hat{g}_n} e^{-\pi n^2/N} = \langle f, g\rangle - \sum_{n\in\mathbb{Z}} \hat{f}_n\overline{\hat{g}_n}(1-e^{-\pi n^2/N}).$$

As $0 \le 1 - e^{-\pi n^2/N} \le \pi n^2/N$ one has by the Cauchy-Schwarz inequality

$$\Big|\sum_{n\in\mathbb{Z}} \hat{f}_n\hat{g}_n(1-e^{-\pi n^2/N})\Big| \le \frac{1}{4\pi N}\sum_{n\in\mathbb{Z}} |\hat{f}_n||\hat{g}_n|(2\pi n)^2 \le \frac{1}{4\pi N}\|f'\|_0\|g'\|_0$$

and the claimed estimate follows.
(iv) By (ii) (for $f = g$ and $k = \ell$) $\|\psi_f^{N,k}\|^2 = \sum_{n\in\mathbb{Z}} |\hat{f}_n|^2 e^{-\pi n^2/N} \le \|f\|_0^2$. $\square$

Next we describe how $T_N^{\alpha,\beta}$ acts on $\psi_f^{N,k}$. This result will be an important ingredient to obtain the asymptotics of the eigenvalues of $Q_N^{\alpha,\beta}$ at the edges. For $f \in L^2(\mathbb{T})$ and $0 \le \ell \le 2N-1$, introduce, with $\alpha_2(x) := \alpha(2x)$,

$$D_\ell^{\alpha,\beta}(f) := 2\cos\big(\frac{\ell\pi}{N} - \frac{i}{2N}\partial_x\big) f + \frac{1}{4N^2}\big(\beta_2(x) + 2\alpha_2(x)\cos\big(\frac{\ell\pi}{N} - \frac{i}{2N}\partial_x\big)\big) f. \tag{4.6}$$

This expression is to be understood in the sense of functional calculus. More precisely, $\cos\big(\frac{\ell\pi}{N} - \frac{i}{2N}\partial_x\big)$ is viewed as a multiplier operator in Fourier space

$$\cos\big(\ell\pi/N - i/2N\partial_x\big) f = \sum_{n\in\mathbb{Z}} \hat{f}_n \cos\big(\ell\pi/N + 2\pi n/2N\big) e^{i2\pi n x}.$$

Theorem 4.4 *For any $f \in C^2$ and $0 \le \ell \le 2N-1$*

$$\|T_N^{\alpha,\beta}\psi_f^{N,\ell} - \psi_{D_\ell^{\alpha,\beta}(f)}^{N,\ell}\| \le \frac{1}{N^3} K_{\alpha,\beta}\|f\|_{C^2} \text{ with } K_{\alpha,\beta} := \|\alpha\|_{C^2} + \|\beta\|_{C^2} + 1.$$

To prove Theorem 4.4 we first need to establish some auxiliary results. First note that $Q_N^{\alpha,\beta} = Q_N^{0,\beta} + \big(Q_N^{0,\alpha} - Q_N^{0,0}\big)Q_N^+ + Q_N^-\big(Q_N^{0,\alpha} - Q_N^{0,0}\big)$ where

$$Q_N^+ = \begin{pmatrix} 0 & 1 & & 0 \\ 0 & 0 & & \vdots \\ \vdots & \vdots & \dots & 1 \\ 1 & 0 & & 0 \end{pmatrix}$$

and Q_N^- is the transpose of Q_N^+. Denote by $T_N^\pm$ the operator on $\mathcal{H}_{2N}$ whose matrix representation with respect to the basis $[\theta_{2N-1}^N, \dots, \theta_0^N]$ are $Q_N^\pm$. Notice that $T_N^\pm$ are isometries as $Q_N^\pm$ are the matrix representations of permutations. Further $T_N^{0,0} = T_N^+ + T_N^-$ as $Q_N^{0,0} = Q_N^+ + Q_N^-$. For any $f \in L^2(\mathbb{T})$ and $0 \le \ell \le 2N-1$, define

$$D_\ell^\pm(f) = \exp\big(\pm i\frac{2\pi\ell - i\partial/\partial x}{2N}\big) f \quad \text{and} \quad D_\ell^{0,0} = D_\ell^+ + D_\ell^-.$$

Lemma 4.5 *For any $f \in L^2(\mathbb{T})$ and $0 \le \ell \le 2N-1$,*

$$T_N^\pm \psi_f^{N,\ell} = \psi_{D_\ell^\pm(f)}^{N,\ell} \quad \text{and} \quad T_N^{0,0}\psi_f^{N,\ell} = \psi_{D_\ell^{0,0}(f)}^{N,\ell}.$$

Proof: Since $\psi_f^{N,\ell}$ is linear in f it suffices to verify the claimed identities for $f(x) = e_k(x) := e^{i2\pi kx}$. By Lemma 4.3 (i) and the fact that $(e^{i\pi nj/N})_{0\le j\le 2N-1}$ is an eigenvector of $Q_N^\pm$ with eigenvalue $e^{\pm i\pi n/N}$ one has

$$T_N^\pm \psi_{e_k}^{N,\ell} = e^{\pm i\pi(k+\ell)/N}\psi_{e_k}^{N,\ell} = \psi_{e^{\pm i\pi(k+\ell)/N}e_k}^{N,\ell}$$

As $e^{\pm i\pi(k+\ell)/N}e^{i2\pi kx} = e^{\pm i\pi\ell/N}e^{\pm\frac{-i\partial/\partial x}{2N}}\big(e^{i2\pi kx}\big) = D_\ell^\pm\big(e^{i2\pi kx}\big)$ the claimed identities follow. $\square$

The key result used in the proof of Theorem 4.4 is the following one.

Lemma 4.6 *Let $f, g \in C^2(\mathbb{T})$. Then with $f_2(x) := f(2x)$, one has*

$$|(T_N^{0,f} - T_N^{0,0})\psi_g^{N,k} - \frac{1}{4N^2}\psi_{gf_2}^{N,k}\| \le \frac{1}{32\pi N^3}\big(\|(gf_2)''\|_{C^0} + \|f\|_{C^0}\|g''\|_{C^0}\big).$$

Proof: Note that $Q_N^{0,f} - Q_N^{0,0}$ is a diagonal matrix with entries $(2N)^{-2}f(j/N)$, $1 \le j \le 2N$. By the definition of $T_N^{0,f}$, it then follows that

$$\big(T_N^{0,f} - T_N^{0,0}\big)\theta_j^N = (2N)^{-2}f\big((2N-j)/N\big)\theta_j^N = (2N)^{-2}f\big(-j/N\big)\theta_j^N.$$

Hence by Lemma 4.3 (i)

$$4N^2\big(T_N^{0,f} - T_N^{0,0}\big)\psi_g^{N,k}(z) = \frac{1}{\sqrt{2N}}\sum_{j=0}^{2N-1} f_2(-\frac{j}{2N})\theta_j^N(z)e^{-i\pi kj/N}\big(e^{-\frac{\Delta}{8\pi N}}g\big)(-\frac{j}{2N}).$$

Furthermore, one has $\psi_{gf_2}^{N,k}(z) = \frac{1}{\sqrt{2N}}\sum_{j=0}^{2N-1}\theta_j^N(z)e^{-i\pi kj/N}e^{-\Delta/8\pi N}(gf_2)\big(-j/2N\big)$. As $(\theta_j)_{0\le j\le 2N-1}$ is an orthonormal basis of $\mathcal{H}_{2N}$, it then follows

$$\|\psi_{f_2g}^{N,k} - 4N^2(T_N^{0,f} - T_N^{0,0})\psi_g^{N,k}\|^2 \le \frac{1}{2N}\sum_{j=1}^{2N-1}\big|\,[e^{-\Delta/8\pi N}, M_{f_2}]g\big(-\frac{j}{2N}\big)\,\big|^2$$

where M_{f_2} denotes the operator on $L^2(\mathbb{T})$ of multiplication by f_2 and $[\cdot,\cdot]$ is the commutator of operators. Hence

$$\|\psi_{f_2g}^{N,k} - 4N^2(T_N^{0,f} - T_N^{0,0})\psi_g^{N,k}\| \le \sup_{0\le x\le 1}\big|\,[e^{-\Delta/8\pi N}, M_{f_2}]g(x)\,\big|.$$

We estimate the latter expression using $e^{-\Delta t} = Id - \Delta\int_0^t e^{-\Delta s}ds$,

$$f_2(x)(e^{-\Delta/8\pi N}g)(x) = f_2(x)g(x) - f_2(x)\left(\int_0^{(8\pi N)^{-1}} e^{-\Delta s}ds\ \Delta g\right)(x).$$

Using the formula of the heat kernel on $\mathbb{R}$ and the identity $\int_{-\infty}^{\infty} e^{-(x-y)^2/4s} dy = \sqrt{4\pi s}$ one gets

$$\big|\big(\int_0^{(8\pi N)^{-1}} e^{-\Delta s} ds\ \Delta g\big)(x)\big| \le \|g''\|_{C^0}(8\pi N)^{-1} \quad \text{and}$$

$$\big|\int_0^{(8\pi N)^{-1}} e^{-\Delta s} ds \Delta(gf_2)(x)\big| \le \|(gf_2)''\|_{C^0}\ (8\pi N)^{-1}.$$

Combining these estimates yields the claimed estimate. □

Finally, for the proof of Theorem 4.4 we will also need the following lemma.

Lemma 4.7 *For $f \in C^1(\mathbb{T})$, denote by M_{f_2} the multiplication operator on $L^2(\mathbb{T})$ by $f_2(x) := f(2x)$. Then the operator $[D_k^{\pm}, M_{f_2}]$ on $L^2(\mathbb{T})$ satisfies*

$$[D_k^{\pm}, M_{f_2}] = \big(f(2x) - f(2x \pm \frac{1}{N})\big) D_k^{\pm} \quad \textit{and} \quad \|[D_k^{\pm}, M_{f_2}]\|_{L^2 \to L^2} \le \frac{1}{N}\|f'\|_{C^0}.$$

Moreover

$$\big\|[T_N^{0,f} - T_N^{0,0}, T_N^{\pm}]\big\|_{\mathcal{H}_{2N} \to \mathcal{H}_{2N}} \le \frac{1}{N}\|f'\|_{C^0}.$$

Proof: Recalling that $D_k^{\pm} = e^{\pm i \frac{2\pi k - i\partial/\partial x}{2N}}$, it is straightforward to verify that the values of the two operators coincide for any $g = e^{i2\pi n x}$. The claimed identity then follows by linearity. The claimed bound of the operator norm of $[D_k^{\pm}, M_{f_2}]$ then follows from the unitarity of $D_k^{\pm}$. The second estimate is proved using the matrix representation of the operators involved. Recall that $Q_N^{0,f} - Q_N^{0,0} = \operatorname{diag}\big((2N)^{-2} f(j/N)_{1 \le j \le 2N}\big)$. Thus $[(Q_N^{0,f} - Q_N^{0,0}), Q_N^{+}]$ is the $2N \times 2N$ matrix

$$\begin{pmatrix} 0 & \gamma_1 & & 0 \\ 0 & 0 & \dots & \vdots \\ \vdots & \vdots & & \gamma_{2N-1} \\ \gamma_{2N} & 0 & & 0 \end{pmatrix} \quad \text{with} \quad \gamma_i = f\big(\frac{i}{N}\big) - f\big(\frac{i+1}{N}\big).$$

Hence

$$\big\|[(Q_N^{0,f} - Q_N^{0,0}), Q_N^{+}]\big\|_{\mathbb{R}^{2N} \to \mathbb{R}^{2N}} = \sup_{1 \le i \le 2N} \big| f\big(\frac{i}{N}\big) - f\big(\frac{i+1}{N}\big)\big| \le \frac{1}{N}\|f'\|_{C^0}.$$

As $Q_N^{-}(Q_N^{0,f} - Q_N^{0,0})$ is the transpose of $(Q_N^{0,f} - Q_N^{0,0})Q_N^{+}$, the same estimate holds for $[(Q_N^{0,f} - Q_N^{0,0}), Q_N^{-}]$. □

Proof of Theorem 4.4: We write $T_N^{\alpha,\beta}$ as a sum of operators

$$\begin{aligned} T_N^{\alpha,\beta} &= T_N^{0,0} + \big(T_N^{0,\beta} - T_N^{0,0}\big) + \big(T_N^{0,\alpha} - T_N^{0,0}\big)T_N^+ + T_N^-\big(T_N^{0,\alpha} - T_N^{0,0}\big) \\ &= T_N^{0,0} + \big(T_N^{0,\beta} - T_N^{0,0}\big) + T_N^{0,0}\big(T_N^{0,\alpha} - T_N^{0,0}\big) + \Big[\big(T_N^{0,\alpha} - T_N^{0,0}\big), T_N^+\Big]\,. \end{aligned}$$

By Lemma 4.5 and Lemma 4.6 we get, respectively, $T_N^{0,0}\psi_f^{N,\ell} = \psi^{N,\ell}_{D_\ell^{0,0}(f)}$ and

$$\big\|\big(T_N^{0,\beta} - T_N^{0,0}\big)\psi_f^{N,\ell} - \frac{1}{4N^2}\psi^{N,\ell}_{\beta_2 f}\big\| \leq \frac{1}{8N^3}\|\beta\|_{C^2}\|f\|_{C^2} \qquad \text{and}$$

$$T_N^{0,0}\big(T_N^{0,\alpha} - T_N^{0,0}\big)\psi_f^{N,\ell} - \frac{1}{4N^2}\psi^{N,\ell}_{D_\ell^{0,0}(\alpha_2 f)} = T_N^{0,0}\big(\big(T_N^{0,\alpha} - T_N^{0,0}\big)\psi_f^{N,\ell} - \frac{1}{4N^2}\psi^{N,\ell}_{\alpha_2 f}\big).$$

As $T_N^{0,0} = T_N^+ + T_N^-$ and $T_N^\pm$ are isometries it follows from Lemma 4.6 that

$$\big\|T_N^{0,0}\big(T_N^{0,\alpha} - T_N^{0,0}\big)\psi_f^{N,\ell} - \frac{1}{4N^2}\psi^{N,\ell}_{D_\ell^{0,0}(\alpha_2 f)}\big\| \leq \frac{1}{4N^3}\|\alpha\|_{C^2}\|f\|_{C^2}.$$

By Lemma 4.7 and Lemma 4.3 (iv) it follows that

$$\big\|\big[\big(T_N^{0,\alpha} - T_N^{0,0}\big), T_N^+\big]\psi_f^{N,\ell}\big\| \leq \frac{\|\alpha'\|_{C^0}}{4N^3}\|\psi_f^{N,\ell}\| \leq \frac{\|\alpha'\|_{C^0}}{4N^3}\|f\|_0.$$

Finally, we need to estimate $\psi^{N,\ell}_{\alpha_2 D_\ell^{0,0} f} - \psi^{N,\ell}_{D_\ell^{0,0}(\alpha_2 f)}$. As by Lemma 4.3 (iv), the linear map $g \mapsto \psi_g^{N,\ell}$ is bounded by 1 on $L^2(\mathbb{T})$, it follows from Lemma 4.7 that

$$\begin{aligned} \big\|\psi^{N,\ell}_{\alpha_2 D_\ell^{0,0} f} - \psi^{N,\ell}_{D_\ell^{0,0}(\alpha_2 f)}\big\| &\leq \|\alpha_2 D_\ell^+ f - D_\ell^+(\alpha_2 f)\|_0 + \|\alpha_2 D_\ell^- f - D_\ell^-(\alpha_2 f)\|_0 \\ &\leq \frac{2}{N}\|\alpha'\|_{C^0}\|f\|_0. \end{aligned}$$

Taking into account the simple identity, $D_\ell^{\alpha,\beta}(f) = D_\ell^{0,0}(f) + \frac{1}{4N^2}\beta_2 f + \frac{1}{4N^2}\alpha_2 D_\ell^{0,0}(f)$, the obtained estimates imply the claimed one. $\Box$

5 Spectral results by quasimodes

In this section we prove results on quasimodes used in the proof of Theorem 2.1. Assume that $\mathcal{H}$ is a finite dimensional Hilbert space with inner product $\langle\psi, \phi\rangle$ and induced norm $\|\psi\| = \langle\psi, \psi\rangle^{1/2}$. Further assume that $A : \mathcal{H} \to \mathcal{H}$ is a selfadjoint linear operator.

Proposition 5.1 (i) *Assume that there exist* $\psi \in \mathcal{H}$ *with* $\|\psi\| = 1, \mu \in \mathbb{R}$ *and* $C > 0$ *so that*

$$\|(A-\mu)\psi\| \leq C. \tag{5.1}$$

Then there exists an eigenvalue λ *of* A *so that* $|\lambda - \mu| \leq C$.
(ii) *Assume that there exist two elements* $\psi_\pm \in \mathcal{H}, \|\psi_\pm\| = 1, \mu \in \mathbb{R}, 0 \leq \theta < 1$, *and* $C > 0$ *so that*

$$\|(A-\mu)\psi_\pm\| \leq C \quad \text{ and } \quad \big|\langle \psi_+, \psi_-\rangle\big| \leq \theta.$$

Then for any $D > 8C(1-\theta)^{-1}$, *there exist two eigenvalues* $\lambda_\pm$ *of* A *so that* $|\lambda_\pm - \mu| \leq D$. *If* $\lambda_+ = \lambda_-$, *then the multiplicity of* λ_+ *is at least two.*

Proof: (i) Denote by $(\lambda_j)_{j\in I}$ the eigenvalues of A listed with their multiplicities. As A is selfadjoint $\mathcal{H}$ has an orthonormal basis of eigenvectors, $(\psi_j)_{j\in I}$, where $\psi_j \in \mathcal{H}$ is an eigenvector corresponding to the eigenvalue λ_j. Assume that for any $j \in I$, $|\lambda_j - \mu| > C$. Then the vector $\psi = \sum_{j\in I}\langle\psi,\psi_j\rangle\psi_j$ satisfies

$$C^2 = C^2\|\psi\|^2 < \sum_{j\in I}\big|\langle\psi,\psi_j\rangle\big|^2(\lambda_j-\mu)^2 = \|(A-\mu)\psi\|^2 \leq C^2,$$

a contradiction. Hence the assumption is not true and (i) follows.
(ii) By item (i), there exists an eigenvalue λ_{i_1} with $|\mu - \lambda_{i_1}| \leq C$. Let us assume that λ_{i_1} has multiplicity one and

$$|\mu - \lambda| > D \quad \forall \lambda \in \mathrm{spec}(A)\setminus\{\lambda_{i_1}\}. \tag{5.2}$$

Then $P := \frac{1}{2\pi}\int_K (z-A)^{-1}dz$ is the orthogonal projector of $\mathcal{H}$ onto the *one* dimensional eigenspace of the eigenvalue λ_{i_1} where K denotes the counterclockwise oriented circle of radius $D/2$ centered at λ_{i_1}. To estimate $P\psi_\pm$ note that $\psi_\pm = (z-A)^{-1}(z-A)\psi_\pm = (z-A)^{-1}(z-\lambda_{i_1})\psi_\pm + (z-A)^{-1}r_\pm$ where $r_\pm = (\lambda_{i_1} - A)\psi_\pm$. Note that $\|r_\pm\| \leq \|(\mu - A)\psi_\pm\| + |\mu - \lambda_{i_1}| \leq 2C$ and

$$(z-A)^{-1}\psi_\pm = (z-\lambda_{i_1})^{-1}\psi_\pm - (z-\lambda_{i_1})^{-1}(z-A)^{-1}r_\pm. \tag{5.3}$$

Write $r_\pm$ as $r_\pm = Pr_\pm + (Id-P)r_\pm$ and use $(z-A)^{-1}Pr_\pm = (z-\lambda_{i_1})^{-1}Pr_\pm$ to see that $\frac{1}{2\pi i}\int_K (z-\lambda_{i_1})^{-1}(z-A)^{-1}Pr_\pm dz = 0$. Hence

$$\frac{1}{2\pi i}\int_K (z-\lambda_{i_1})^{-1}(z-A)^{-1}r_\pm dz = \frac{1}{2\pi i}\int_K (z-\lambda_{i_1})^{-1}(z-A)^{-1}(Id-P)r_\pm dz.$$

By Cauchy's theorem we then get

$$\frac{1}{2\pi i}\int_K (z-\lambda_{i_1})^{-1}(z-A)^{-1}r_\pm dz = (\lambda_{i_1} - A)^{-1}(Id-P)r_\pm. \tag{5.4}$$

Hence integrating (5.3) along the contour K one concludes from (5.4) that

$$P\psi_\pm = \psi_\pm + (\lambda_{i_1} - A)^{-1}(Id - P)r_\pm.$$

By (5.1) - (5.2) we then have $\|(\lambda_{i_1} - A)^{-1}(Id - P)r_\pm\| \le D^{-1}2C$ and thus, with $\eta := 2CD^{-1} < 1$, it follows that $0 \le 1 - \|P\psi_\pm\|^2 \le \eta^2$, i.e.

$$\|P\psi_\pm\| \ge \sqrt{1-\eta^2} > 1 - \eta, \tag{5.5}$$

and $\left|\langle P\psi_+, P\psi_-\rangle - \langle \psi_+, \psi_-\rangle\right| \le 2\eta + \eta^2$, implying that

$$\left|\langle P\psi_+, P\psi_-\rangle\right| \le \theta + 2\eta + \eta^2. \tag{5.6}$$

In order to assure that $P\psi_+$ and $P\psi_-$ are linearly independent we request that $\left|\langle P\psi_+, P\psi_-\rangle\right| < \|P\psi_+\|\|P\psi_-\|$. In view of (5.5) and (5.6) this latter inequality is satisfied when $0 < \eta < \frac{1-\theta}{4}$. But by the definition of η and D, one has $2C\eta^{-1} = D > \frac{8C}{1-\theta}$. Thus we proved that $P\psi_+$ and $P\psi_-$ are linearly independent, contradicting our assumption. Hence there are at least two (counted with multiplicities) eigenvalues of A inside the circle of radius D and center λ_{i_1}. $\square$

6 Quasimodes for the bulk of $\mathrm{spec}(T_N^{\alpha,\beta})$

We want to apply Proposition 5.1 (ii) to the bulk of the spectrum of $T_N^{\alpha,\beta}$,

$$\{\lambda_{2\ell-1}^N, \lambda_{2\ell}^N \mid M < \ell < N - M\}$$

where $M \equiv M_N = [F(N)]$. For $M < \ell < N - M$ and $N \ge 3$ arbitrary choose $\mu \equiv \mu_\ell^N$ to be the ℓ'th double eigenvalue of $T_N^{0,0}$, $\mu_\ell^N := -2\cos\frac{\ell\pi}{N}$. Our construction of quasimodes follows the standard procedure of perturbation theory of double eigenvalues: first we construct two approximate eigenvectors $\psi_{0,\pm}^\ell$ of the operator $\prod_\ell \circ (T_N^{\alpha,\beta} - T_N^{0,0})\big|_{E_\ell}$ where E_ℓ denotes the two dimensional eigenspace of the eigenvalue $-2\cos\frac{\ell\pi}{N}$ of $T_N^{0,0}$ and the operator is the composition of the restriction of the perturbation $T_N^{\alpha,\beta} - T_N^{0,0}$ to E_ℓ with the orthogonal projection $\prod_\ell$ onto E_ℓ. The two quasimodes $\psi_\pm^\ell$ are then obtained by adding a first order correction to $\psi_{0,\pm}^\ell$. To this aim introduce $\widetilde{\psi}_+^\ell = \psi^{N,N+\ell}$ and $\widetilde{\psi}_-^\ell = \psi^{N,N-\ell}$ where we recall that $\psi^{N,k}$ denotes the eigenvector of $T_N^{0,0}$ with eigenvalue $2\cos\frac{k\pi}{N}$ defined by (4.1). One has

$$\widetilde{\psi}_\pm^\ell = (2N)^{-1/2}\sum_{n=0}^{2N-1} e^{\mp in\pi\frac{N-\ell}{N}}\theta_n^N$$

$$T_N^{0,0}\widetilde{\psi}_\pm^\ell = -2\cos\frac{\ell\pi}{N}\cdot\widetilde{\psi}_\pm^\ell \quad\text{and}\quad \langle\widetilde{\psi}_+^\ell, \widetilde{\psi}_-^\ell\rangle = 0.$$

Denote by $\hat{\alpha}_k, \hat{\beta}_k$, $k \in \mathbb{Z}$ the Fourier coefficients of α, β and set

$$\hat{\gamma}_\ell := \hat{\beta}_\ell - 2\cos\frac{\ell\pi}{N}\hat{\alpha}_\ell, \qquad e^{-i\eta_\ell} := \hat{\gamma}_\ell/|\hat{\gamma}_\ell| \text{ if } \hat{\gamma}_\ell \neq 0, \quad \text{and} \quad e^{i\eta_\ell} := 1 \text{ if } \hat{\gamma}_\ell = 0.$$

For any $M < \ell < N - M$, let $\psi_\pm^\ell := \psi_{0,\pm}^\ell + \varphi_\pm^\ell$ where

$$\psi_{0,\pm}^\ell := \frac{\widetilde{\psi}_+^\ell \pm e^{i\eta_\ell}\widetilde{\psi}_-^\ell}{\sqrt{2}} \quad \text{and} \quad \varphi_\pm^\ell := -\sum_{n \neq N \pm \ell} \frac{\langle \psi^{N,n}, (T_N^{\alpha,\beta} - T_N^{0,0})\psi_{0,\pm}^\ell \rangle}{2\cos\frac{\ell\pi}{N} + 2\cos\frac{n\pi}{N}} \psi^{N,n}.$$

Lemma 6.1 *The elements $\psi_\pm^\ell$, $M < \ell < N - M$, of $\mathcal{H}_{2N}$ satisfy*

(i) $\langle \psi_+^\ell, \psi_-^\ell \rangle = O(\frac{K_{\alpha,\beta}^2}{M^2})$ *and* $\|\psi_\pm^\ell\| = 1 + O(\frac{K_{\alpha,\beta}^2}{M^2})$;

(ii) $\big\|(T_N^{\alpha,\beta} + 2\cos\frac{\ell\pi}{N})\psi_\pm^\ell\big\| = O\left(K_{\alpha,\beta}^2 \frac{1}{N^2 M}\right)$ *where* $K_{\alpha,\beta} = \|\alpha\|_{C^2} + \|\beta\|_{C^2} + 1$.

First we need to establish the following auxiliary result.

Lemma 6.2 *(i) For any $M < \ell < N - M$ and $n \neq N \pm \ell$*

$$|\langle \psi^{N,n}, (T_N^{\alpha,\beta} - T_N^{0,0})\psi_{0,\pm}^\ell \rangle| = O\left(\frac{K_{\alpha,\beta}}{N^2}\left(min_\pm \frac{1}{(N-n\pm\ell)^2} + \frac{1}{N}\right)\right).$$

(ii) For any $M < \ell < N - M$, $\langle \psi_{0,+}^\ell, (T_N^{\alpha,\beta} - T_N^{0,0})\psi_{0,-}^\ell \rangle = O\left(\frac{K_{\alpha,\beta}}{N^3}\right)$ and

$$\langle \psi_{0,+}^\ell, (T_N^{\alpha,\beta} - T_N^{0,0})\psi_{0,+}^\ell \rangle = \frac{e^{-2\pi\ell^2/N}}{2N^2}\Re\hat{\gamma}_\ell + O\left(\frac{K_{\alpha,\beta}}{N^3}\right),$$

$$\langle \psi_{0,-}^\ell, (T_N^{\alpha,\beta} - T_N^{0,0})\psi_{0,-}^\ell \rangle = -\frac{e^{-2\pi\ell^2/N}}{2N^2}\Re\hat{\gamma}_\ell + O\left(\frac{K_{\alpha,\beta}}{N^3}\right).$$

Proof: By (4.1), $\psi_{0,\pm}^\ell = \frac{\psi^{N,N+\ell} \pm e^{i\eta_l}\psi^{N,N-l}}{\sqrt{2}}$. Recall that by (4.3), one has for $f \equiv 1$ the identity $\psi^{N,k} = \psi_1^{N,k}$ and by Proposition 4.4, for f arbitrary, $T_N^{\alpha,\beta}\psi_f^{N,k} = \psi_{D_k^{\alpha,\beta}f}^{N,k} + O(\frac{K_{\alpha,\beta}}{N^3})$ where $D_k^{\alpha,\beta}$ is given by

$$2\cos\big(k\pi/N - i(2N)^{-1}\partial_x\big) + (2N)^{-2}\big(\beta_2(x) + 2\alpha_2(x)\cos\big(k\pi/N - i(2N)^{-1}\partial_x\big)\big).$$

For $f \equiv 1$ one has $D_k^{\alpha,\beta}1 = 2\cos\big(k\pi/N\big) + g(x)$ where $g(x) = (2N)^{-2}\big(\beta_2(x) + 2\alpha_2(x)\cos\big(k\pi/N\big)\big)$ and

$$(T_N^{\alpha,\beta} - T_N^{0,0})\psi^{N,k} = (T_N^{\alpha,\beta} - T_N^{0,0})\psi_1^{N,k} = \psi_g^{N,k} + O\left(\frac{K_{\alpha,\beta}}{N^3}\right).$$

By Lemma 4.3(ii) we have

$$\langle \psi_1^{N,n}, \psi_g^{N,k} \rangle = (2N)^{-2} \left(\widehat{(\beta_2)}_{k-n} + 2\cos\left(k\pi/N\right) \widehat{(\alpha_2)}_{k-n} \right) e^{-\pi(n-k)^2/2N}.$$

Choosing $k = N \pm \ell$, item (i) then follows as by assumption $\alpha, \beta \in C_0^2(\mathbb{T}, \mathbb{R})$. To prove item (ii) note that for $n = N \pm \ell$, one has $N \pm \ell - k \in \{0, \pm 2\ell\}$. It then follows from the definition of $e^{i\eta_\ell}$ that $\langle \psi_{0,+}^\ell, \psi_g^{N,N+\ell} - e^{i\eta_\ell} \psi_g^{N,N-\ell} \rangle = 0$ and

$$\langle \psi_{0,+}^\ell, \psi_g^{N,N+\ell} + e^{i\eta_\ell} \psi_g^{N,N-\ell} \rangle = e^{i\eta_\ell} \left(\widehat{(\beta_2)}_{2\ell} - 2\cos\left(\ell\pi/N\right) \widehat{(\alpha_2)}_{2\ell} \right) \frac{e^{-2\pi\ell^2/N}}{4N^2}$$

$$+ e^{-i\eta_\ell} \left(\widehat{(\beta_2)}_{-2\ell} - 2\cos\left(\ell\pi/N\right) \widehat{(\alpha_2)}_{-2\ell} \right) \frac{e^{-2\pi\ell^2/N}}{4N^2} = \frac{e^{-2\pi\ell^2/N}}{4N^2} 2\Re\hat{\gamma}_\ell.$$

From these and similar computations the claimed estimates follow. □

Proof of Lemma 6.1: (i) First note that for any $M < \ell < N - M$ and $0 \le n \le N$ with $n \ne N \pm \ell$, $|2\cos\frac{\ell\pi}{N} + 2\cos\frac{n\pi}{N}| \ge \frac{2M\pi}{N^2}$. Indeed, in the case where $M < \ell \le N/2$ and $0 \le k := N - n < \ell$ one has

$$|2\cos\frac{\ell\pi}{N} - 2\cos\frac{k\pi}{N}| = 2\int_{\frac{k\pi}{N}}^{\frac{\ell\pi}{N}} \sin(x)dx \ge 2\int_{\frac{k\pi}{N}}^{\frac{\ell\pi}{N}} \frac{2x}{\pi} dx$$

leading to the claimed lower bound. All other cases are treated in a similar way. By Lemma 6.2 (i) one then concludes that

$$\|\varphi_\pm^\ell\| = O(K_{\alpha,\beta}\frac{1}{M}). \tag{6.1}$$

On the other hand, $\psi_{0,+}^\ell$ and $\psi_{0,-}^\ell$ are orthogonal to each other, and both are orthogonal to $\varphi_\pm^\ell$. Hence $\langle \psi_+^\ell, \psi_-^\ell \rangle = \langle \varphi_+^\ell, \varphi_-^\ell \rangle$. Combined with the above estimate one gets $\langle \psi_+^\ell, \psi_-^\ell \rangle = O(\frac{K_{\alpha,\beta}^2}{M^2})$. Using in addition that $\|\psi_{0,\pm}^\ell\| = 1$ one then has $\|\psi_\pm^\ell\| = 1 + O(\frac{K_{\alpha,\beta}^2}{M^2})$. (ii) We apply standard perturbation theory and write

$$T_N^{\alpha,\beta} \psi_\pm^\ell = (T_N^{0,0} + (T_N^{\alpha,\beta} - T_N^{0,0}))(\psi_{0,\pm}^\ell + \varphi_\pm^\ell) \tag{6.2}$$

and split the right hand side of (6.2) into four parts

$$T_N^{0,0}\psi_{0,\pm}^\ell, \quad T_N^{0,0}\varphi_\pm^l, \quad (T_N^{\alpha,\beta} - T_N^{0,0}))\psi_{0,\pm}^\ell, \quad (T_N^{\alpha,\beta} - T_N^{0,0}))\varphi_\pm^\ell.$$

Note that $T_N^{0,0}\psi_{0,\pm}^\ell = -2\cos\frac{\ell\pi}{N}\psi_{0,\pm}^\ell$ and

$$T_N^{0,0}\varphi_\pm^l = -\sum_{n \ne N\pm\ell} 2\cos\frac{n\pi}{N} \cdot \frac{\langle \psi^{N,n}, (T_N^{\alpha,\beta} - T_N^{0,0})\psi_{0,\pm}^\ell \rangle}{2\cos\frac{\ell\pi}{N} + 2\cos\frac{n\pi}{N}} \psi^{N,n}$$

$$= - \sum_{n \neq N \pm \ell} \left(1 - \frac{2\cos\frac{\ell\pi}{N}}{2\cos\frac{\ell\pi}{N} + 2\cos\frac{n\pi}{N}} \right) \langle \psi^{N,n}, (T_N^{\alpha,\beta} - T_N^{0,0}) \psi_{0,\pm}^{\ell} \rangle \psi^{N,n}.$$

In view of the definition of $\varphi_{\pm}^{\ell}$, this yields the identity

$$T_N^{0,0} \varphi_{\pm}^{\ell} = -2\cos\frac{\ell\pi}{N} \varphi_{\pm}^{\ell} - \sum_{n \neq N \pm \ell} \langle \psi^{N,n}, (T_N^{\alpha,\beta} - T_N^{0,0}) \psi_{0,\pm}^{\ell} \rangle \psi^{N,n}.$$

Combined with

$$(T_N^{\alpha,\beta} - T_N^{0,0}) \psi_{0,\pm}^{\ell} = \sum_{n \neq N \pm \ell} \langle \psi^{N,n}, (T_N^{\alpha,\beta} - T_N^{0,0}) \psi_{0,\pm}^{\ell} \rangle \psi^{N,n} + \sum_{s \in \{+,-\}} \langle \psi_{0,s}^{\ell}, (T_N^{\alpha,\beta} - T_N^{0,0}) \psi_{0,\pm}^{\ell} \rangle \psi_{0,s}^{\ell}$$

one gets

$$(T_N^{\alpha,\beta} - T_N^{0,0}) \psi_{0,\pm}^{\ell} + T_N^{0,0} \varphi_{\pm}^{\ell} = -2\cos\frac{\ell\pi}{N} \varphi_{\pm}^{\ell} + \sum_{s \in \{+,-\}} \langle \psi_{0,s}^{\ell}, (T_N^{\alpha,\beta} - T_N^{0,0}) \psi_{0,\pm}^{\ell} \rangle \psi_{0,s}^{\ell}.$$

By Lemma 6.2 (ii) it then follows that

$$(T_N^{\alpha,\beta} - T_N^{0,0}) \psi_{0,\pm}^{\ell} + T_N^{0,0} \varphi_{\pm}^{\ell} = -2\cos\frac{\ell\pi}{N} \varphi_{\pm}^{\ell} + O\left(\frac{K_{\alpha,\beta}}{N^2} \left(\frac{1}{M^2} + \frac{1}{N} \right) \right).$$

Finally, the expression

$$(T_N^{\alpha,\beta} - T_N^{0,0}) \varphi_{\pm}^{\ell} = - \sum_{n \neq N \pm \ell} \frac{\langle \psi^{N,n}, (T_N^{\alpha,\beta} - T_N^{0,0}) \psi_{0,\pm}^{\ell} \rangle}{2\cos\frac{\ell\pi}{N} + 2\cos\frac{n\pi}{N}} (T_N^{\alpha,\beta} - T_N^{0,0}) \psi^{N,n}$$

can be estimated by Lemma 6.2 (i) to get for some constant $C \geq 1$,

$$\|(T_N^{\alpha,\beta} - T_N^{0,0}) \varphi_{\pm}^{\ell}\| \leq C \sum_{n \neq N \pm \ell} \frac{N^2}{M} \frac{K_{\alpha,\beta}}{N^2} \left(\min_{\pm} \frac{1}{(N - n \pm \ell)^2} + \frac{1}{N} \right) \|(T_N^{\alpha,\beta} - T_N^{0,0}) \psi^{N,n}\|.$$

Inspecting the proof of Lemma 6.2 (i) one sees that $\|(T_N^{\alpha,\beta} - T_N^{0,0}) \psi^{N,n}\| = O(\frac{K_{\alpha,\beta}}{N^2})$, yielding $\|(T_N^{\alpha,\beta} - T_N^{0,0}) \varphi_{\pm}^{\ell}\| = O(\frac{K_{\alpha,\beta}^2}{N^2 M})$. Combining all the above estimates, item (ii) follows. □

Lemma 6.1 allows to apply Proposition 5.1 and leads to the following result.

Proposition 6.3 *For any $N \geq 3$ and $M < \ell < N - M$, there exists a pair of eigenvalues $\tau_-^{N,\ell} \leq \tau_+^{N,\ell}$ of $T_N^{\alpha,\beta}$ satisfying*

$$\left| \tau_{\pm}^{N,\ell} + 2\cos\frac{\ell\pi}{N} \right| = O\left(K_{\alpha,\beta} \frac{1}{N^2 M} \right) \quad \textit{where} \quad K_{\alpha,\beta} = \|\alpha\|_{C^2} + \|\beta\|_{C^2} + 1.$$

For N sufficiently large these pairs are separated from each other,

$$\cdots < \tau_-^{N,\ell} \leq \tau_+^{N,\ell} < \tau_-^{N,\ell+1} \leq \tau_+^{N,\ell+1} < \cdots$$

Proof: According to Lemma 6.1, for any $N \geq 3$ and any $M < \ell < N - M$

$$\|(T_N^{\alpha,\beta} + 2\cos\frac{\ell\pi}{N})\psi_\pm^N\| = O\left(\frac{K_{\alpha,\beta}^2}{N^2 M}\right).$$

By Proposition 5.1 (ii), there are two eigenvalues $\tau_-^{N,\ell} \leq \tau_+^{N,\ell}$ of $T_N^{\alpha,\beta}$ satisfying

$$|\,\tau_\pm^{N,\ell} + 2\cos\frac{\ell\pi}{N}\,| = O\left(\frac{K_{\alpha,\beta}^2}{N^2 M}\right).$$

In case $\tau_+^{N,\ell} = \tau_-^{N,\ell}$, the eigenvalue has multiplicity at least two. To see that for N sufficiently large, one has $\tau_+^{N,\ell} < \tau_-^{N,\ell+1}$, recall from the proof of Lemma 6.1 that

$$|2\cos\frac{\ell\pi}{N} - 2\cos\frac{(\ell+1)\pi}{N}| \geq \frac{2M\pi}{N^2} \qquad M < \ell < N - M.$$

Hence by choosing N_0 sufficiently large, the pairs of eigenvalues $\tau_\pm^{N,\ell}$ with $N \geq N_0$ satisfy $\tau_-^{N,M+1} \leq \tau_+^{N,M+1} < \tau_-^{N,M+2} \leq \tau_+^{N,M+2} < \cdots < \tau_-^{N,N-1-M} \leq \tau_+^{N,N-1-M}$.
□

7 Quasimodes for the edges of $\mathrm{spec}(T_N^{\alpha,\beta})$

In this section we want to apply Proposition 5.1 to the two edges of the spectrum of $T_N^{\alpha,\beta}$. They are treated in the same way, so we concentrate on the left edge only,

$$\lambda_0^N < \lambda_1^N \leq \lambda_2^N < \ldots < \lambda_{2M-1}^N \leq \lambda_{2M}^N$$

where again $M \equiv M_N = [F(N)]$. For $0 \leq j \leq 2M$, choose as approximate eigenvalue

$$\mu_-^{N,j} = -2 + \frac{1}{4N^2}\lambda_j^- \tag{7.1}$$

where $\lambda_0^- < \lambda_1^- \leq \lambda_2^- < \ldots$ are the periodic eigenvalues of $H_- = -d^2/dx^2 + q_-$, considered on the interval $[0, 1]$. Here

$$q_- = \beta_2 - 2\alpha_2 \quad \text{and} \quad \alpha_2(x) = \alpha(2x),\ \beta_2(x) = \beta(2x).$$

Furthermore choose as quasimodes

$$\varphi_-^{N,j}(z) := \psi_{g_j^-}^{N,N}(z) = (4N)^{-1/4}\int_0^1 g_j^-(s)\varrho_N(z, \frac{1}{2} + is)e^{-2N\pi s^2}ds \tag{7.2}$$

where $(g_j^-)_{j\geq 0}$ is an orthonormal basis of eigenfunctions of H_-. First we need to establish bounds for g_j^- and its derivatives. By the counting lemma (cf e.g. [22]), for any N with

$$M_N > 2(1+\|q_-\|_0)e^{\|q_-\|_0} \tag{7.3}$$

it follows that for $0 \leq j \leq 2M$,

$$|\lambda_j^-| \leq 4\pi^2(M+\frac{1}{2})^2 \leq 8\pi^2 F(N)^2. \tag{7.4}$$

Recall that $K_{\alpha,\beta} = \|\alpha\|_{C^2} + \|\beta\|_{C^2} + 1$ for any $\alpha, \beta \in C^2(\mathbb{T})$.

Lemma 7.1 *For any N satisfying* (7.3) *and any $\alpha, \beta \in C^2(\mathbb{T})$,*

(i) $\|(g_j^-)'\|_0 \leq (2K_{\alpha,\beta} + 8\pi^2 F(N)^2)^{1/2}$; (ii) $\|(g_j^-)''\|_0 \leq 2K_{\alpha,\beta} + 8\pi^2 F(N)^2$;

(iii) $\|(g_j^-)'''\|_0 \leq (2K_{\alpha,\beta} + 8\pi^2 F(N)^2)^{3/2} + 2K_{\alpha,\beta}$;

(iv) $\|(g_j^-)^{IV}\|_0 \leq 3(2K_{\alpha,\beta} + 8\pi^2 F(N)^2)^2 + 2K_{\alpha,\beta} \leq 4(2K_{\alpha,\beta} + 8\pi^2 F(N)^2)^2$.

Proof: (i) Taking the inner product of $-(g_j^-)'' + q_- g_j^- = \lambda_j^- g_j^-$ with g_j^-, integrating by parts and using (7.4) and $\|g_j^-\|_0 = 1$ yields the bound (i) for $\|(g_j^-)'\|_0$. (ii) Using again $(g_j^-)'' = q_- g_j^- - \lambda_j^- g_j^-$ one gets

$$\|(g_j^-)''\|_0 \leq \|q_-\|_0 \|g_j^-\|_0 + |\lambda_j^-| \|g_j^-\|_0 \leq 2K_{\alpha\beta} + 8\pi^2 F(N)^2.$$

(iii) is obtained by deriving $(g_j^-)'' = q_- g_j^- - \lambda_j g_j^-$ and using (i). (iv) is obtained by arguing in the same way. □

We also need bounds for $\|g_j^-\|_{C^0}$ and $\|g_j^-\|_{C^2}$. It is convenient to formulate the result in a general form. For a real valued potential $q \in L^2(\mathbb{T})$, denote by $(f_j)_{j\geq 0}$ an orthonormal basis of periodic eigenfunctions of $H = -d^2/dx^2 + q$ on $[0,1]$.

Lemma 7.2 (i) *The expression $\sup_{j\geq 0} \|f_j\|_{C^0}$ is bounded uniformly on bounded sets of potentials in $L^2(\mathbb{T})$.*
(ii) *For any N with $M = [F(N)] > 2(1+\|q\|_0)e^{\|q\|_0}$ and any $0 \leq j \leq 2M$*

$$\|f_j''\|_{C^0} \leq \big(\|q\|_0 + 8\pi^2 F(N)^2\big)\|f_j\|_{C^0} \text{ and } \|f_j'\|_{C^0} \leq 2\big(\|q\|_{C^0} + 8\pi^2 F(N)^2\big).$$

Proof: (i) It is well known that f_0 doesn't vanish on $[0,1]$. As for any $j \geq 1$, f_j is orthogonal to f_0, it has to vanish at least once. Hence there exists $0 \leq x_j < 1$ so that $f_j(x_j) = 0$. As a consequence, the translate $T_{x_j} f_j = f_j(\cdot + x_j)$ is a Dirichlet eigenfunction for the translated potential $T_{x_j} q$. Note that $\|T_{x_j} q\|_0 = \|q\|_0$ and $\|T_{x_j} f_j\|_{C^0} = \|f_j\|_{C^0}$. Therefore $\sup_{j\geq 1} \|f_j\|_{C^0}$ is bounded uniformly on bounded sets of potentials in $L^2(\mathbb{T})$ by the corresponding result for the Dirichlet problem – see e.g. [28, p. 35]. It remains to bound $\|f_0\|_{C^0}$. As $\lambda_0(q)$ is never a Dirichlet eigenvalue, one has

$$f_0(x) = \frac{1}{c_0}\big(y_1(x,\lambda_0) + \frac{1-y_1(1,\lambda_0)}{y_2(1,\lambda_0)}y_2(x,\lambda_0)\big) \text{ where}$$

$$c_0 = c_0(q) = \Big(\int_0^1 \big(y_1(x,\lambda_0) + \frac{1-y_1(1,\lambda_0)}{y_2(1,\lambda_0)}y_2(x,\lambda_0)\big)^2 dx\Big)^{1/2}$$

and y_1, y_2 are the fundamental solutions of $-y'' + qy = \lambda y$. By [28, p. 7]

$$|y_i(x,\lambda_0(q),q)| \le e^{\|q\|_0} \quad \forall \quad 0 \le x \le 1 \text{ and } i = 1,2.$$

Further, by [28, p 18], $y_i(x,\lambda,q)$ is a compact function of $q \in L^2(\mathbb{T})$, uniformly on bounded subsets of $[0,1]\times\mathbb{C}$. By [22, p 199], $L^2(\mathbb{T}) \to \mathbb{R}, q \mapsto \lambda_0(q)$ is a compact function as well and so is $q \mapsto y_2(1,\lambda_0(q),q)$. As $\lambda_0(q)$ is never a Dirichlet eigenvalue $y_2(1,\lambda_0(q),q) > 0$ for any q in $L^2(\mathbb{T})$. By the compactness, $y_2(1,\lambda_0(q),q)$ is uniformly bounded away from 0 on bounded sets of potentials in $L^2(\mathbb{T})$. Similarly, one argues by compactness to conclude that $c_0(q) > 0$ is uniformly bounded away from 0 on bounded sets of potentials in $L^2(\mathbb{T})$.
(ii) Note that $\|f_j''\|_{C^0} \le \big(\|q\|_{C^0} + |\lambda_j|\big)\|f_j\|_{C^0}$. Hence the claimed estimate of $\|f_j''\|_{C^0}$ follows from item (i) and (7.3) - (7.4). Finally, for any $0 \le x, y \le 1$, $f_j'(x) = f_j'(y) + \int_y^x f_j''(s)ds$. Integrate in y and apply the Cauchy-Schwarz inequality to conclude that

$$\|f_j'\|_{C^0} \le \|f_j'\|_0 + \|f_j''\|_0 \le \big(\|q\|_{C^0} + 8\pi^2 F(N)^2\big)^{1/2} + \big(\|q\|_{C^0} + 8\pi^2 F(N)^2\big)$$

where the latter inequality follows from the proof of Lemma 7.1 (i), (ii). □

Lemma 7.3 *For any N with $M = [F(N)] > 2(1+\|q_-\|_0)e^{\|q_-\|_0}$ and any $0 \le j \le 2M$, the elements $\varphi_-^{N,j}$ in $\mathcal{H}_{2N}$ satisfy the following estimates:*

(i) $|\langle\varphi_-^{N,j}, \varphi_-^{N,k}\rangle - \delta_{j,k}| \le \frac{1}{4\pi N}(2K_{\alpha,\beta} + 8\pi^2 F(N)^2) \quad \forall 0 \le k \le 2M.$

(ii) $\|(T_N^{\alpha,\beta} + 2 - \frac{1}{4N^2}\lambda_j^-)\varphi_-^{N,j}\| \le \frac{F(N)^2}{N^3}(K_{\alpha,\beta}+1)^2 C$

where $C > 0$ can be chosen uniformly on L^2-bounded subsets of $C^2(\mathbb{T})$.

Proof: (i) By the definition (4.3) and Lemma 4.3 (iii), $|\langle\varphi_-^{N,j}, \varphi_-^{N,k}\rangle - \langle g_j^-, g_k^-\rangle| \le \frac{1}{4\pi N}\|(g_j^-)'\|_0\|(g_k^-)'\|_0$. By Lemma 7.1 (i), we get $\|(g_j^-)'\|_0\|(g_k^-)'\|_0 \le 2K_{\alpha,\beta} + 8\pi^2 F_1(N)^2$ and hence the claimed estimate. (ii) By the triangle inequality

$$\|T_N^{\alpha,\beta}\varphi_-^{N,j} - \mu_-^{N,j}\varphi_-^{N,j}\| \le \|T_N^{\alpha,\beta}\varphi_-^{N,j} - \psi^{N,N}_{D_N^{\alpha,\beta}(g_j^-)}\| + \|\psi^{N,N}_{D_N^{\alpha,\beta}(g_j^-)} - \mu_-^{N,j}\varphi_-^{N,j}\|. \tag{7.5}$$

Let us begin by estimating the latter term. By definition, $\varphi_-^{N,j} = \psi_{g_j^-}^{N,N}$ and hence $\mu_-^{N,j}\varphi_-^{N,j} = \psi_{\mu_-^{N,j} g_j^-}^{N,N}$. By Lemma 4.3 (iv) we then conclude

$$\|\psi^{N,N}_{D_N^{\alpha,\beta}(g_j^-)} - \mu_-^{N,j}\varphi_-^{N,j}\| \leq \|D_N^{\alpha,\beta}(g_j^-) - \mu_-^{N,j} g_j^-\|_0. \tag{7.6}$$

As $D_\ell^{\alpha,\beta} = 2\cos\big(\frac{\ell\pi}{N} - \frac{i}{2N}\partial_x\big) + \frac{1}{4N^2}\big(\beta_2 + \alpha_2 2\cos(\frac{\ell\pi}{N} - \frac{i}{2N}\partial_x)\big)$ one gets for $\ell = N$ $D_N^{\alpha,\beta} - 2\cos\big(-\frac{i}{2N}\partial_x\big) + \frac{1}{4N^2}\big(\beta_2 - \alpha_2 2\cos(-\frac{i}{2N}\partial_x)\big)$. Furthermore, $\mu_-^{N,j} g_j^- = \big(-2 - \frac{1}{4N^2}\partial_x^2\big)g_j^- + \frac{1}{4N^2}\big(\beta_2 - 2\alpha_2\big)g_j^-$. Hence we get

$$\begin{aligned}\|D_N^{\alpha,\beta}(g_j^-) - (\mu_-^{N,j})g_j^-\|_0 \leq \|2\cos\big(-\frac{i}{2N}\partial_x\big)g_j^- - \big(2 + \frac{1}{4N^2}\partial_x^2\big)g_j^-\|_0 \\ + \frac{1}{4N^2}\|2\alpha_2\big(1 - \cos(-\frac{i}{2N}\partial_x)\big)g_j^-\|_0.\end{aligned} \tag{7.7}$$

The latter two terms are estimated individually.

$$2\cos\big(-\frac{i}{2N}\partial_x\big)g_j^- = \sum_{n\in\mathbb{Z}} 2\cos\big(\frac{\pi n}{N}\big)\big(\widehat{g_j^-}\big)_n e^{i2\pi nx}.$$

Using the Taylor expansion of $2\cos\frac{\pi n}{N}$, one concludes that

$$\begin{aligned}&\|2\cos\big(-\frac{i}{2N}\partial_x\big)g_j^- - \big(2 + \frac{1}{4N^2}\partial_x^2\big)g_j^-\|_0 \\ &\leq \frac{1}{12}\frac{1}{(2N)^4}\Big(\sum_{n\in\mathbb{Z}} \big|(\widehat{g_j^-})_n\big|^2 (2\pi n)^8\Big)^{1/2} \leq \frac{1}{12}\frac{1}{(2N)^4}\|(g_j^-)^{IV}\|_0.\end{aligned}$$

By Lemma 7.1, it then follows that

$$\Big\|2\cos\big(-\frac{i}{2N}\partial_x\big)g_j^- - \big(2 + \frac{1}{4N^2}\partial_x^2\big)g_j^-\Big\|_0 \leq \frac{1}{48}\frac{1}{N^4}\big(2K_{\alpha,\beta} + 8\pi^2 F(N)^2\big)^2. \tag{7.8}$$

In a similar way one estimates

$$\begin{aligned}&\frac{1}{4N^2}\Big\|2\alpha_2\big(1 - \cos(-\frac{i}{2N}\partial_x)g_j^-\big)\Big\|_0 \leq \frac{\|\alpha\|_{C^0}}{4N^2}\Big(\sum_{n\in\mathbb{Z}}\big(\frac{\pi n}{N}\big)^4 \big|(\widehat{g_j^-})_n\big|^2\Big)^{1/2} \\ &\leq \frac{\|\alpha\|_{C^0}}{(2N)^4}\|(g_j^-)''\|_0 \leq \frac{K_{\alpha,\beta}}{16N^4}\big(2K_{\alpha,\beta} + 8\pi^2 F(N)^2\big) \leq \frac{1}{32N^4}\big(2K_{\alpha,\beta} + 8\pi^2 F(N)^2\big)^2\end{aligned}$$

where for the latter inequality we again used Lemma 7.1. Combining (7.6) - (7.8) yields

$$\Big\|\psi^{N,N}_{D_N^{\alpha,\beta}(g_j^-)} - \mu_-^{N,j}\varphi_-^{N,j}\Big\| \leq \frac{F(N)^4}{16N^4}(8\pi^2 + 2K_{\alpha,\beta})^2. \tag{7.9}$$

It remains to estimate the first term on the right hand side of (7.5). By Theorem 4.4 (with $\ell = N$),

$$\big\|T_N^{\alpha,\beta}\varphi_-^{N,j} - \psi^{N,N}_{D_N^{\alpha,\beta}(g_j^-)}\big\| \le \frac{1}{N^3} K_{\alpha,\beta}\|g_j^-\|_{C^2}. \tag{7.10}$$

By Lemma 7.2, for any $0 \le j \le 2M_N$

$$\|g_j^-\|_{C^2} \le \big(2K_{\alpha,\beta} + 8\pi^2 F(N)^2\big)C \tag{7.11}$$

where $C \ge 1$ can be chosen uniformly on bounded subsets of $L^2(\mathbb{T})$. Combining (7.5), (7.9), (7.10), and (7.11) yields $\big\|T_N^{\alpha,\beta}\varphi_-^{N,j} - \mu_-^{N,j}\varphi_-^{N,j}\big\| \le \frac{F(N)^4}{16N^4}(8\pi^2 + 2K_{\alpha,\beta})^2 + \frac{F(N)^2}{N^3}K_{\alpha,\beta}(2K_{\alpha,\beta} + 8\pi^2)C$ where $C \ge 1$ can be chosen uniformly on bounded subsets of $L^2(\mathbb{T})$. $\square$

Proposition 7.4 *For any $N \ge 3$ and any $0 \le j \le 2M$ there exists an eigenvalue $\tau_-^{N,j}$ of $T_N^{\alpha,\beta}$ satisfying*

$$\big|\,\tau_-^{N,j} - \mu_-^{N,j}\,\big| \le \frac{F(N)^2}{N^3}(K_{\alpha,\beta}+1)^2 C$$

where $C > 0$ can be chosen uniformly on L^2-bounded subsets of $C^2(\mathbb{T})$. For N sufficiently large, the eigenvalues $\tau_-^{N,j}$ can be listed in increasing order

$$\tau_-^{N,0} < \tau_-^{N,1} \le \tau_-^{N,2} < \ldots < \tau_-^{N,2M-1} \le \tau_N^{N,2M}.$$

Proof: According to Lemma 7.3, for any $N \ge 3$ and $0 \le j,k \le 2M$,

$$\big\|\big(T_N^{\alpha,\beta} - \mu_-^{N,j}\big)\varphi_-^{N,j}\big\| \le C_N := \frac{F(N)^2}{N^3}(K_{\alpha,\beta}+1)^2 C$$

where C can be chosen uniformly on L^2-bounded subsets of $C^2(\mathbb{T})$. Furthermore, for N_0 sufficiently large, $\big|\langle\varphi_-^{N,j}, \varphi_-^{N,k}\rangle - \delta_{j,k}\big| < \frac{1}{2} \quad \forall N \ge N_0$. We now apply Proposition 5.1 (i) or (ii) depending on whether $\mu_-^{N,j}$ is sufficiently isolated or not. Note the the pair $\mu_-^{N,2\ell}, \mu_-^{N,2\ell-1}$ is separated from $\{\mu_-^{N,j}\,\big|\,0 \le j \le 2M\}\backslash\{\mu_-^{N,2\ell}, \mu_-^{N,2\ell-1}\}$ by at least $O(N^{-2})$, uniformly on L^2-bounded sets of α's and β's. If $\mu_-^{N,2\ell} - \mu_-^{N,2\ell-1} \le 2C_N$, then

$$\begin{aligned}\big\|\big(T_N^{\alpha,\beta} - \mu_-^{N,2\ell}\big)\varphi_-^{N,2\ell-1}\big\| &\le C_N + \big|\,\mu_-^{N,2\ell} - \mu_-^{N,2\ell-1}\,\big|\,\big\|\varphi_-^{N,2\ell-1}\big\| \\ &\le C_N + 2C_N\big(1 + O(F(N)^2N^{-1})\big)\end{aligned}$$

Applying Proposition 5.1 (ii) to $\mu_-^{N,2\ell}, \varphi_-^{N,2\ell}, \varphi_-^{N,2\ell-1}$ and $D_N = 2 \cdot 8 \cdot 3C_N$ we conclude that there are two eigenvalues $\tau_-^{N,2\ell-1} \leq \tau_-^{N,2\ell}$ of $T_N^{\alpha,\beta}$ so that

$$\left| \tau_-^{N,j} - \mu_-^{N,2\ell} \right| \leq D_N = \frac{F(N)^2}{N^3}(K_{\alpha,\beta}+1)^2 C$$

for $j \in \{2\ell, 2\ell-1\}$, where C can be chosen uniformly on L^2-bounded subsets of $C^2(\mathbb{T})$. If $\mu_-^{N,2\ell} - \mu_-^{N,2\ell-1} > 2C_N$, then apply Proposition 5.1 (i) to conclude that for $j \in \{2\ell, 2\ell-1\}$, there exists an eigenvalue $\tau_-^{N,j}$ of $T_N^{\alpha,\beta}$ so that $|\tau_-^{N,j} - \mu_-^{N,j}| \leq C_N$. In particular, we then conclude that $\tau_-^{N,2\ell-1} < \tau_-^{N,2\ell}$.
Recall that the pairs $\mu_-^{N,2\ell}, \mu_-^{N,2\ell-1}$ are separated from each other by $O(N^{-2})$. As $F(N) \leq N^\eta$ with $0 < \eta < 1/2$ it then follows from the definition of C_N that for N sufficiently large $\tau_-^{N,0} < \tau_-^{N,1} \leq \tau_-^{N,2} < \ldots < \tau_-^{N,2N-1} \leq \tau_-^{N,2M}$. $\square$

8 Asymptotics of the periodic eigenvalues

The aim of this section is to prove Theorem 2.1 stated in the introduction.
Proof of Theorem 2.1 In view of Proposition 6.3, Proposition 7.4, and the result corresponding to Proposition 7.4 for the right edge of the spectrum we have obtained three groups of eigenvalues. At the left and right edge of spec$(T_N^{\alpha,\beta})$ there are according to Proposition 7.4, $2M+1$ eigenvalues, which for N sufficiently large are different from each other when counted with multiplicities. In the bulk of spec$(T_N^{\alpha,\beta})$, we found according to Proposition 6.3, $N-M-1$ pairs of eigenvalues of $T_N^{\alpha,\beta}$ which for N sufficiently large are again different from each other. It remains to show that

$$\tau^{N,2M} < \tau^{N,2M+1} \quad \text{and} \quad \tau^{N,2N-2M-2} < \tau^{N,2N-2M-1}.$$

To see it, note that by the Taylor expansion of cos and (7.4), we have

$$\mu_-^{N,2M} = -2 + \frac{1}{N^2}\lambda_{2M}^- \leq -2 + \pi^2 \frac{(M+1/2)^2}{N^2}.$$

Hence $-2\cos\frac{(M+1)\pi}{N} - \mu_-^{N,2M} \geq \frac{M\pi^2}{N^2} + O\big(\frac{1}{N^2}\big)$. Moreover, by Proposition 7.4, $\tau^{N,2M} - \mu_-^{N,2M} = O\big(\frac{M^2}{N^3}\big)$ and $\tau^{N,2M+1} + 2\cos\frac{(M+1)\pi}{N} = O\big(\frac{1}{N^2}\big)$ by Proposition 6.3. Therefore, for N sufficiently large, $\tau^{N,2M} < \tau^{N,2M+1}$. Similarly one shows that $\tau^{N,2N-2M-2} < \tau^{N,2N-2M-1}$. Hence the eigenvalues $(\tau^{N,n})_{0\leq n\leq 2N-1}$ of $T_N^{\alpha,\beta}$ are listed in increasing order (and with multiplicities) and thus coincide with $(\lambda_n^N)_{0\leq n\leq 2N-1}$. The claimed estimates now follow from Proposition 6.3 and Proposition 7.4. $\square$

To finish this section, let us mention that for smooth potentials and with some effort, our method allows to compute the full asymptotic expansion in $\frac{1}{N^2}$ of all the eigenvalues of $T_N^{\alpha,\beta}$. For eigenvalues in the bulk, such an asymptotic expansion is obtained by regular perturbation theory at any order. As the eigenvalues come in separated pairs and the eigenvalues forming such a pair might coincide, their asymptotics are obtained via a $2 \times 2-$block diagonalization and a subsequent straightforward diagonalization of the (symmetric) $2\times 2-$blocks. For eigenvalues in one of the edges of the spectrum, the asymptotics is obtained by adding corrections to the 'densities' $g_j^{\pm}$ in (7.2), obtained by improving Theorem 4.4 so that the remainder term can be chosen to be of arbitrary order in N^{-2}. These corrections can be explicitly computed by solving homological equations, obtained from the asymptotic expansion of the operator $D_\ell^{\alpha,\beta}$ in (4.6) and solved by inverting certain Hill operators. As the eigenvalues of a Hill operator come in separated pairs and two eigenvalues forming such a pair might coincide, one obtains their asymptotics also via a $2 \times 2-$block diagonalization.

9 Asymptotics of the discriminant

The principal goal of this section is to prove Theorem 2.3 concerning the asymptotics of the discriminant $\Delta_N(\mu)$. Recall (cf. [19], Section 2) that $\Delta_N^2(\mu) - 4$ is related to the characteristic polynomial of $Q_N^{\alpha,\beta}$ as follows

$$\Delta_N(\mu)^2 - 4 = q_N^{-2} \prod_{j=0}^{2N-1} (\lambda_j^N - \mu). \tag{9.1}$$

First we derive asymptotics of $q_N = \prod_1^N (1 + \frac{1}{4N^2}\alpha(\frac{n}{N}))$. For later reference we derive at the same time also asymptotics for $p_N := \frac{1}{2}tr(Q_N^{\alpha,\beta}) = \frac{1}{2}\sum_0^{2N-1} \lambda_n^N = \frac{1}{4N^2}\sum_1^N \beta(i/N)$. It turns out that the asymptotics of p_N is better than one could expect from the asymptotics of the eigenvalues in Theorem 2.1.

Proposition 9.1 *Uniformly on bounded subsets of functions α, β in $C_0^2(\mathbb{T})$,*

$$q_N = 1 + O(N^{-3}) \quad \textit{and} \quad p_N = O(N^{-3}). \tag{9.2}$$

Proof: As $\int_0^1 \beta(x)dx = 0$ it follows from (A.2) that

$$p_N = \frac{1}{2}tr(Q_N^{\alpha,\beta}) = (2N)^{-2} \sum_{i=1}^N \beta(\frac{i}{N}) = O(N^{-3}).$$

Similarly, one has $\sum_{i=1}^{N}(2N)^{-2}\alpha(\frac{i}{N}) = O(N^{-3})$ and thus

$$q_N = \exp\left(\sum_{i=1}^{N} \log\left(1 + \frac{1}{4N^2}\alpha(\frac{i}{N})\right)\right) = \exp(O(N^{-3}))$$

leading to the claimed estimate $q_N = 1 + O(N^{-3})$. $\square$

In the introduction we have also introduced the discriminants $\Delta_\pm$. For $q_\pm = 0$ one gets $\Delta(\lambda) = 2\cos(\sqrt{\lambda}/2)$ (cf end of Appendix A) and hence, with $\pi_n = \pi n$ for $n \geq 1$,

$$\Delta(\lambda)^2 - 4 = -4\sin^2(\sqrt{\lambda}/2) = -\lambda \prod_{n\geq 1} \frac{(4n^2\pi^2 - \lambda)^2}{16\pi_n^4}.$$

Similarly (see [22]), for $q_\pm$ arbitrary, one gets the following product representation

$$\Delta_\pm(\lambda)^2 - 4 = (\lambda_0^\pm - \lambda) \prod_{n\geq 1} \frac{(\lambda_{2n}^\pm - \lambda)(\lambda_{2n-1}^\pm - \lambda)}{16\pi_n^4}. \tag{9.3}$$

Finally recall from the introduction that $\Lambda^{\pm,M}$ denote the boxes

$$\Lambda^{\pm,M} \equiv \Lambda_2^{\pm,M} := [\lambda_0^\pm - 2, \lambda_{2[F(M)]}^\pm + 2] + i[-2, 2]$$

where $M = [F(N)]$ and $N \geq N_0$. We chose $N_0 \in \mathbb{Z}_{\geq 1}$ so that

$$\lambda_{2k+1}^\pm - \lambda_{2k}^\pm \geq 6 \quad \forall\, k \geq F(F(N_0)). \tag{9.4}$$

The estimates (2.8) and (2.9) of Theorem 2.3 are obtained in a similar fashion so we concentrate on the proof of (2.8) only. We first need to establish several auxiliary results. We cover $\Lambda^{-,M}$ by open neighborhoods, each containing one spectral band and its adjacent gaps,

$$\Lambda^{-,M} = \bigcup_{n\leq [F(M)]} \Lambda_{n,\rho}^-$$

where $\Lambda_{1,\rho}^- := [\lambda_0^- - 3, \lambda_2^- + 2\rho] + i[-3, 3]$ and for $2 \leq n \leq [F(M)] - 1$,

$$\Lambda_{n,\rho}^- := [\lambda_{2n-3}^- - 2\rho, \lambda_{2n}^- + 2\rho] + i[-3, 3] \qquad \text{and}$$

$$\Lambda_{[F(M)],\rho}^- := [\lambda_{2[F(M)]-3}^- - 2\rho, \lambda_{2[F(M)]}^- + 3] + i[-3, 3]$$

with $\rho > 0$ chosen so that

$$\lambda_{2k}^\pm + 2\rho < \lambda_{2k+1}^\pm - 2\rho \quad \forall\, k \geq 0. \tag{9.5}$$

We will study the asymptotics of $\Delta_N^2\big(-2+\frac{1}{4N^2}\lambda\big)$ in each domain $\Lambda^-_{n,\rho}$ separately. For this purpose introduce

$$P_1^N(\mu) := \varepsilon^{-3}\Pi_{j=0}^{2}(\lambda_j^N-\mu) \quad \text{and} \quad P_n^N(\mu) := \varepsilon^{-4}\Pi_{j=2n-3}^{2n}(\lambda_j^N-\mu) \;\; \forall 2\le n\le M$$

where $\varepsilon = 1/4N^{-2}$. Furthermore define for $1\le n\le M$, $Q_n^N(\mu)$

$$\Delta_N^2(\mu)-4 = \varepsilon^4 P_n^N(\mu)Q_n^N(\mu).$$

Defining $\pi_k = k\pi$ for $k\neq 0$ and $\pi_0=1$, we write similarly $\Delta_-^2(\lambda)-4 = \frac{1}{16\pi^4}P_1^-(\lambda)Q_1^-(\lambda)$ with

$$P_1^-(\lambda) = \Pi_{0\le j\le 2}(\lambda_j^- -\lambda) \quad \text{and} \quad Q_1^-(\lambda) = \Pi_{k\ge 2}\frac{(\lambda_{2k}^- -\lambda)(\lambda_{2k-1}^- -\lambda)}{16\pi_k^4}$$

whereas for $2\le n\le M$, we define $\Delta_-^2(\lambda)-4 = \frac{1}{16\pi_{n-1}^4}\frac{1}{4\pi_n^2}P_n^-(\lambda)Q_n^-(\lambda)$ with

$$P_n^-(\lambda) := (\lambda_{2n}^- -\lambda)(\lambda_{2n-1}^- -\lambda)(\lambda_{2n-2}^- -\lambda)(\lambda_{2n-3}^- -\lambda)$$

and

$$Q_n^-(\lambda) = \frac{1}{4\pi_n^2}(\lambda_0^- -\lambda)\prod_{k\neq n,n-1}\frac{(\lambda_{2k}^- -\lambda)(\lambda_{2k-1}^- -\lambda)}{16\pi_k^4}.$$

By Theorem 2.1, for λ in $\Lambda^-_{n,\varrho}$ with $2\le n\le M$,

$$\begin{aligned} P_n^N(-2+\frac{1}{4N^2}\lambda) &= \prod_{j=2n-3}^{2n}\Big(\frac{\lambda_j^N-(-2+\varepsilon\lambda_j^-)}{\varepsilon}+(\lambda_j^- -\lambda)\Big) \\ &= \prod_{j=2n-3}^{2n}\big(\lambda_j^- -\lambda+O\big(\frac{M^2}{N}\big)\big) = P_n^-(\lambda)+O\big(n^3\frac{M^2}{N}\big) \end{aligned} \tag{9.6}$$

where we used that $\lambda_j^- -\lambda = O(n)$ for $\lambda\in\Lambda^-_{n,\rho}$ and $2n-3\le j\le 2n$. Similarly , for $n=1$, one has

$$P_1^N(-2+\frac{1}{4N^2}\lambda) = P_1^-(\lambda)+O(\frac{M^2}{N}).$$

In order to prove (2.8) we show that, for $2\le n\le M$,

$$\Delta_N^2(-2+\varepsilon\lambda)-4 = \frac{1}{16\pi_{n-1}^4}\frac{1}{4\pi_n^2}\big(P_n^-(\lambda)+O\big(n^3\frac{M^2}{N}\big)\big)Q_n^-(\lambda)+O\big(\frac{M^2}{N}\big)+O\big(\frac{1}{M}\big) \tag{9.7}$$

uniformly for λ in $\Lambda^-_{n,\varrho}$ and, for $n = 1$,

$$\Delta_N^2(-2+\varepsilon\lambda) - 4 = \frac{1}{16\pi^4}\big(P_1^-(\lambda) + O\big(\frac{M^2}{N}\big)\big)Q_1^-(\lambda) + O\big(\frac{M^2}{N}\big) + O\big(\frac{1}{M}\big) \tag{9.8}$$

uniformly for λ in $\Lambda^-_{1,\varrho}$. The estimates (9.7) and (9.8) are proven in two steps.

Lemma 9.2 *Uniformly for any $\mu = -2 + \varepsilon\lambda$ where $\lambda \in \Lambda^-_{n,\varrho}$ and $1 \le n \le F(M)$*

$$\prod_{j=2M+1}^{2N-2M-2} (\lambda_j^N - \mu) = \frac{N^{4M+2}}{(2\pi)^{4M}(M!)^4}\big(1 + O\big(\frac{n^2}{M}\big)\big).$$

Proof: Set $\xi^N_{2N-1} = 4$ and, for $1 \le \ell \le N-1$,

$$\xi^N_{2\ell} = \xi^N_{2\ell-1} = 2(1 - \cos\frac{\ell\pi}{N}). \tag{9.9}$$

Note that ξ_j^N is an increasing sequence satisfying for $2M+1 \le j \le 2N-2M-2$

$$\xi_j^N \ge \xi^N_{2M+1} > \xi^N_{2M} = 2\big(1 - \cos\frac{M\pi}{N}\big) > \frac{M^2\pi^2}{N^2}\big(1 - \frac{\pi^2}{12}\frac{M^2}{N^2}\big).$$

For $j \in \{2\ell, 2\ell-1\}$ and $\mu = -2 + \varepsilon\lambda$ with $\lambda \in \Lambda^-_{n,\varrho}$, in view of Theorem 2.1,

$$\begin{aligned}\lambda_j^N - \mu &= -2\cos\frac{\ell\pi}{N} + O\big(\frac{1}{F(N)N^2}\big) + 2 - \varepsilon\lambda\\ &= \xi_j^N - \varepsilon\lambda + O\big(\frac{1}{F(N)N^2}\big) = \xi_j^N + O\big(\frac{n^2}{N^2}\big)\end{aligned}$$

where we used that $\lambda = O(n^2)$ for $\lambda \in \Lambda^-_{n,\varrho}$. Hence

$$\begin{aligned}\prod_{j=2M+1}^{2N-2M-2} \big(\lambda_j^N - \mu\big) &= \prod_{j=2M+1}^{2N-2M-2} \xi_j^N\big(1 + \frac{1}{\xi_j^N}O\big(\frac{n^2}{N^2}\big)\big)\\ &= \prod_{j=2M+1}^{2N-2M-2} \xi_j^N \cdot \prod_{j=2M+1}^{2N-2M-2} \big(1 + \frac{1}{\xi_j^N}O\big(\frac{n^2}{N^2}\big)\big).\end{aligned} \tag{9.10}$$

The latter two products are estimated separately. As by (9.9), $\frac{1}{\xi_j^N} = O(\frac{N^2}{M^2})$ for $2M < j < 2N-2M-1$ and hence $\frac{1}{\xi_j^N}O(\frac{n^2}{N^2}) = O(\frac{F(M)^2}{M^2})$ for $1 \le n \le F(M)$ one can estimate

$$\sum_{j=2M+1}^{2N-2M-2} \log\big(1 + \frac{1}{\xi_j^N}O\big(\frac{n^2}{N^2}\big)\big) = O\big(\frac{n^2}{N^2}\big)\sum_{\ell=M+1}^{N-M-1} \frac{1}{1 - \cos\frac{\ell\pi}{N}}.$$

Note that $\frac{1}{1-\cos x}$ is a monotonically decreasing function on $[\frac{M\pi}{N}, \pi]$. Hence

$$\frac{\pi}{N} \sum_{\ell=M+1}^{N-M-1} \frac{1}{1-\cos\frac{\ell\pi}{N}} \leq \int_{\frac{M\pi}{N}}^{\pi-\frac{M\pi}{N}} \frac{dx}{1-\cos x}.$$

Taking into account that $1-\cos x = 2\sin^2\frac{x}{2}$ and making the change of variable of integration $t := \frac{x}{2}$ we get

$$\int_{\frac{M\pi}{N}}^{\pi-\frac{M\pi}{N}} \frac{dx}{1-\cos x} = \int_{\frac{M\pi}{N}}^{\pi-\frac{M\pi}{N}} \frac{dx}{2\sin^2\frac{x}{2}} = \int_{\frac{M\pi}{2N}}^{\frac{1}{2}\left(\pi-\frac{M\pi}{N}\right)} \frac{dt}{\sin^2 t} = -\frac{\cos t}{\sin t} \Big|_{\frac{M\pi}{2N}}^{\frac{1}{2}(\pi-\frac{M\pi}{N})} \leq \frac{N}{M}.$$

Thus $\sum_{j=2M+1}^{2N-2M-2} \log\left(1+\frac{1}{\xi_j^N}O\left(\frac{n^2}{N^2}\right)\right) = O\left(\frac{n^2}{M}\right)$ leading to

$$\prod_{j=2M+1}^{2N-2M-2} \left(1+\frac{1}{\xi_j^N}O\left(\frac{n^2}{N^2}\right)\right) = e^{O\left(\frac{n^2}{M}\right)} = 1+O\left(\frac{n^2}{M}\right).$$

It then follows that

$$\prod_{j=2M+1}^{2N-2M-2} (\lambda_j^N - \mu) = \left(1+O\left(\frac{n^2}{M}\right)\right) \cdot \prod_{j=2M+1}^{2N-2M-2} \xi_j^N$$

$$= 2^{2(N-2M-1)} \prod_{\ell=M+1}^{N-M-1} \left(1-\cos\left(\frac{\ell\pi}{N}\right)\right)^2 \cdot \left(1+O\left(\frac{n^2}{M}\right)\right)$$

$$= 2^{2(N-2M-1)} \left(\frac{\prod_{\ell=1}^{N-1}\left(1-\cos\frac{\ell\pi}{N}\right)}{\prod_{\ell=1}^{M}\left(1-\cos\frac{\ell\pi}{N}\right)\prod_{\ell=1}^{M}\left(1-\cos\frac{(N-\ell)\pi}{N}\right)}\right)^2 \left(1+O\left(\frac{n^2}{M}\right)\right).$$

By Lemma A.1, Lemma A.3, and Lemma A.4 this latter expression can be estimated by

$$2^{2N-4M-2} \frac{(2N2^{-N})^2\left(1+O\left(\frac{1}{N}\right)\right)}{\left(\frac{\pi^2}{2N^2}\right)^{2M}(M!)^4\left(1-O\left(\frac{M^3}{N^2}\right)\right)2^{2M}\left(1-O\left(\frac{M^3}{N^2}\right)\right)} \left(1+O\left(\frac{n^2}{M}\right)\right)$$

$$= \frac{1}{2^{4M}} \frac{N^{4M+2}}{\pi^{4M}(M!)^4} \left(1+O\left(\frac{n^2}{M}\right)\right)$$

which is the claimed estimate. □

Lemma 9.3 *Uniformly for $\mu = -2 + \varepsilon\lambda$ with λ in $\Lambda^-_{n,\varrho}$ and $1 \le n \le M$,*

$$\prod_{j=2N-2M-1}^{2N-1} (\lambda_j^N - \mu) = 2^{4M+2}\big(1 + O(\frac{M^3}{N^2})\big).$$

Proof: In view of Theorem 2.1 (right edge), for any $0 \le j \le 2M,\ \lambda \in \Lambda^-_{n,\varrho}$,

$$\lambda^N_{2N-1-j} - \mu = 2 - \varepsilon\lambda_j^+ + O\big(\frac{M^2}{N^3}\big) - (-2 + \varepsilon\lambda) = 4 + O\big(\frac{M^2}{N^2}\big)$$

as $\lambda_j^+ = O(M^2)$ for any $0 \le j \le 2M$. Thus

$$\prod_{j=2N-2M-1}^{2N-1} (\lambda_j^N - \mu) = \prod_{j=0}^{2M} (\lambda^N_{2N-1-j} - \mu) = 4^{2M+1} \prod_{j=0}^{2M} \big(1 + O\big(\frac{M^2}{N^2}\big)\big)$$
$$= 2^{4M+2} \exp\big(\sum_{j=0}^{2M} \log\big(1 + O\big(\frac{M^2}{N^2}\big)\big)\big).$$

Using the bound $\sum_{j=0}^{2M} \log\big(1 + O\big(\frac{M^2}{N^2}\big)\big) = O\big(\frac{M^3}{N^2}\big)$ we get

$$\prod_{j=2N-2M-1}^{2N-1} (\lambda_j^N - \mu) = 2^{4M+2} \exp\big(O\big(\frac{M^3}{N^2}\big)\big) = 2^{4M+2}\big(1 + O\big(\frac{M^3}{N^2}\big)\big)$$

which is the claimed estimate. $\square$

Lemma 9.4 *Uniformly for $\mu = -2 + \varepsilon\lambda$ with $\lambda \in \Lambda^-_{n,\varrho}$, the product $\prod_{j=0}^{2M} (\lambda_j^N - \mu)$ satisfies the following estimates: (i) for $2 \le n \le F(M)$*

$$\frac{\pi^{4M}}{4} \frac{(M!)^4}{N^{4M+2}} \frac{1}{16\pi^4_{n-1}} \frac{1}{4\pi_n^2} \big(P_n^-(\lambda) + O(\frac{n^3 M^2}{N})\big) \cdot Q_n^-(\lambda) \cdot (1 + O(\frac{n^2}{M})) \cdot (1 + O(\frac{M^2}{N}));$$

(ii) for $n = 1$

$$\frac{\pi^{4M}}{4} \frac{(M!)^4}{N^{4M+2}} \frac{1}{16\pi^4} \big(P_1^-(\lambda) + O(\frac{M^2}{N})\big) \cdot Q_1^-(\lambda) \cdot (1 + O(\frac{1}{M})) \cdot (1 + O(\frac{M^2}{N})).$$

Proof: In view of the asymptotics of Theorem 2.1, with $\varepsilon = (2N)^{-2}$,

$$\lambda_j^N - \mu = -2 + \varepsilon\lambda_j^- + 2 - \varepsilon\lambda + O(\frac{M^2}{N^3}) = \varepsilon(\lambda_j^- - \lambda + O(\frac{M^2}{N})).$$

Hence $\prod_{j=0}^{2M}(\lambda_j^N - \mu) = \varepsilon^{2M+1}\prod_{j=0}^{2M}(\lambda_j^- - \lambda + O(\frac{M^2}{N}))$. The items (i) and (ii) are proved in a very similar way - in fact (ii) is a little simpler. Hence we concentrate on (i), i.e. the case where $2 \le n \le F(M)$. Given λ in $\Lambda_{n,\varrho}^-$, the latter product is split up into three parts,

$$\prod_{j=0}^{2n-4}(\lambda_j^- - \lambda + O(\frac{M^2}{N}))\cdot\prod_{j=2n-3}^{2n}(\lambda_j^- - \lambda + O(\frac{M^2}{N}))\cdot\prod_{j=2n+1}^{2M}(\lambda_j^- - \lambda + O(\frac{M^2}{N})).$$

Note that

$$\prod_{j=2n-3}^{2n}(\lambda_j^- - \lambda + O(\frac{M^2}{N})) = P_n^-(\lambda) + O(\frac{n^3M^2}{N})$$

uniformly for $\lambda \in \Lambda_{n,\varrho}^-$ with $2 \le n \le M$, cf. equation (9.6). Next consider

$$\prod_{j=0}^{2n-4}\big(\lambda_j^- - \lambda + O(\frac{M^2}{N})\big) = \prod_{j=0}^{2n-4}(\lambda_j^- - \lambda)\cdot\prod_{j=0}^{2n-4}\big(1 + \frac{1}{\lambda_j^- - \lambda}O(\frac{M^2}{N})\big).$$

We claim that

$$\prod_{j=0}^{2n-4}(1 + \frac{1}{\lambda_j^- - \lambda}O(\frac{M^2}{N})) = 1 + O(\frac{M^2}{N})$$

uniformly for λ in $\Lambda_{n,\rho}^-$. Indeed, by the choice of ϱ, the factors $(\lambda - \lambda_j^-)^{-1}$, $2\ell - 1 \le j \le 2\ell$, can be estimated by $O((n^2 - \ell^2)^{-1})$ uniformly for $\lambda \in \Lambda_{n,\varrho}^-$ and $1 \le \ell \le n-2$. Hence $\sum_{j=0}^{2n-4}\frac{1}{\lambda - \lambda_j^-}$ is a bounded analytic function of $\lambda \in \Lambda_{n,\rho}^-$ with a bound depending on ρ. Similarly one treats

$$\prod_{j=2n+1}^{2M}(\lambda_j^- - \lambda + O(\frac{M^2}{N})) = \prod_{j=2n+1}^{2M}(\lambda_j^- - \lambda)\cdot\prod_{j=2n+1}^{2M}(1 + \frac{1}{\lambda_j^- - \lambda}O(\frac{M^2}{N})).$$

Again, by the choice of ϱ and the asymptotics of λ_j^-, $\sum_{j=2n+1}^{2M}\frac{1}{\lambda_j^- - \lambda}$ is a bounded analytic function of $\lambda \in \Lambda_{n,\rho}^-$. Thus

$$\prod_{j=2n+1}^{2M}\big(1 + \frac{1}{\lambda_j^- - \lambda}O(\frac{M^2}{N})\big) = 1 + O(\frac{M^2}{N}).$$

Combining the estimates obtained we have

$$\prod_{j=0}^{2M}(\lambda_j^N - \mu) = \varepsilon^{2M+1} \cdot \prod_{k=1}^{M} 2^4\pi_k^4 \cdot (\lambda_0^- - \lambda) \cdot \prod_{k=1}^{n-2} \frac{(\lambda_{2k}^- - \lambda)(\lambda_{2k-1}^- - \lambda)}{2^4\pi_k^4} \cdot \frac{1}{16\pi_{n-1}^4}$$
$$\cdot \frac{1}{16\pi_n^4} \cdot \big(P_n^-(\lambda) + O(\frac{n^3M^2}{N})\big) \cdot \prod_{k=n+1}^{M} \frac{(\lambda_{2k}^- - \lambda)(\lambda_{2k-1}^- - \lambda)}{2^4\pi_k^4} \cdot (1 + O(\frac{M^2}{N})).$$

Finally we note that, with $\lambda_j^- = 4k^2\pi^2 + \alpha_j$ for $j \in \{2k, 2k-1\}$, where $\alpha_j = O(1)$,

$$\prod_{k=M+1}^{\infty} \frac{(\lambda_{2k}^- - \lambda)(\lambda_{2k-1}^- - \lambda)}{2^4\pi_k^4} = \exp\left\{\sum_{k=M+1}^{\infty} \log\big(\frac{\lambda_{2k}^- - \lambda}{4\pi_k^2}\big) + \log\big(\frac{\lambda_{2k-1}^- - \lambda}{4\pi_k^2}\big)\right\}$$
$$= \exp\left\{\sum_{k=M+1}^{\infty} \log\big(1 + \frac{\alpha_{2k} - \lambda}{4\pi_k^2}\big) + \log\big(1 + \frac{\alpha_{2k-1} - \lambda}{4\pi_k^2}\big)\right\} = 1 + O(\frac{n^2}{M})$$

uniformly for $\lambda \in \Lambda_{n,\rho}^-, 1 \leq n \leq M$. Here we used $\sum_{k=M+1}^{\infty} \frac{1}{k^2} \leq \int_M^\infty \frac{1}{x^2}dx = \frac{1}{M}$. Furthermore $\prod_{k=1}^{M} 2^4\pi_k^4 == (2\pi)^{4M}(M!)^4$. By the definition of $Q_n^-(\lambda)$, we then obtain that $\prod_{j=0}^{2M}(\lambda_j^N - \mu)$ equals

$$\varepsilon^{2M+1}\frac{(2\pi)^{4M}(M!)^4}{16\pi_{n-1}^4 \cdot 4\pi_n^2}\big(P_n^-(\lambda) + O(\frac{n^3M^2}{N})\big)Q_n^-(\lambda)\big(1 + O(\frac{M^2}{N})\big)\big(1 + O(\frac{n^2}{M})\big)$$

as claimed. □

Lemma 9.5 *Uniformly for $\lambda \in \Lambda_{n,\varrho}^-$, $1 \leq n \leq M$, $Q_n^-(\lambda) = O(n^2)$.*

Proof: By the Counting Lemma (cf [22]) for periodic eigenvalues there exits $n_0 \geq 1$ so that $|\lambda_n^\pm - 4n^2\pi^2| \leq 1$ for any $n \geq n_0$. Note that n_0 can be chosen uniformly for bounded sets of functions $\alpha, \beta \in C_0^2$. It turns out that the cases $1 \leq n < n_0$ and $n_0 \leq n \leq M$ have to be treated separately. However they can be proved in a similar way and so we concentrate on the case $n_0 \leq n \leq M$ only.

$$Q_n^-(\lambda) = \frac{1}{4\pi_n^2}(\lambda_0^- - \lambda)\prod_{k \neq n, n-1} \frac{(\lambda_{2k}^- - \lambda)(\lambda_{2k-1}^- - \lambda)}{16\pi_k^4} \tag{9.11}$$

and that $\frac{\sin(\sqrt{\lambda}/2)}{\sqrt{\lambda}/2}$ can be written as an infinite product,

$$\frac{\sin(\sqrt{\lambda}/2)}{\sqrt{\lambda}/2} = \prod_{m \geq 1} \frac{m^2\pi^2 - \lambda/4}{m^2\pi^2} = \prod_{m \geq 1} \frac{4\pi_m^2 - \lambda}{4\pi_m^2}.$$

Hence for $\lambda \in \Lambda^-_{n,\varrho}$,

$$Q^-_n(\lambda) = \frac{\lambda^-_0 - \lambda}{4\pi_n^2} \left(\frac{\sin(\sqrt{\lambda}/2)}{\sqrt{\lambda}/2} \right)^2 \left(\frac{4\pi_n^2 \cdot 4\pi_{n-1}^2}{(4\pi_n^2 - \lambda)(4\pi_{n-1}^2 - \lambda)} \right)^2 f^-_n(\lambda) \tag{9.12}$$

where $f^-_n(\lambda) = \prod_{k \neq n,n-1} \frac{(\lambda^-_{2k} - \lambda)(\lambda^-_{2k-1} - \lambda)}{(4\pi_k^2 - \lambda)^2}$. Clearly, uniformly for $\lambda \in \Lambda^-_{n,\varrho}$, $n_0 \leq n \leq M$, one has $\frac{\lambda - \lambda^-_0}{4\pi_n^2} = 1 + O(\frac{1}{n})$ and

$$\left(\frac{\sin(\sqrt{\lambda}/2)}{\sqrt{\lambda}/2} \right)^2 \left(\frac{4\pi_n^2 \cdot 4\pi_{n-1}^2}{(4\pi_n^2 - \lambda)(4\pi_{n-1}^2 - \lambda)} \right)^2 =$$

$$\left(\frac{\sin(\sqrt{\lambda}/2)}{(\pi_n - \sqrt{\lambda}/2)(\pi_{n-1} - \sqrt{\lambda}/2)} \right)^2 \cdot 4^3 \cdot \frac{\pi_n^4 \pi_{n-1}^4}{\lambda (2\pi_n + \sqrt{\lambda})^2 (2\pi_{n-1} + \sqrt{\lambda})^2} = O(n^2)$$

where we used that for $\lambda \in \Lambda^-_{n,\varrho}$, $n_0 \leq n \leq M$, $\frac{\sin(\sqrt{\lambda}/2)}{(\pi_n - \sqrt{\lambda}/2)(\pi_{n-1} - \sqrt{\lambda}/2)} = O(1)$. Finally we need to estimate $f^-_n(\lambda)$. For $n \geq n_0$, by the choice of $\rho > 0$ there exists $\rho' > 0$ so that $|4\pi_k^2 - \lambda| \geq \frac{1}{\rho'} |k^2 - n^2|$, $\quad \forall k \neq n, n-1, \quad \forall \lambda \in \Lambda^-_{n,\varrho}$. Thus

$$\begin{aligned} \left| \frac{(\lambda^-_{2k} - \lambda)(\lambda^-_{2k-1} - \lambda)}{(4\pi_k^2 - \lambda)^2} \right| &\leq \left(1 + \left| \tfrac{\lambda^-_{2k} - 4\pi_k^2}{4\pi_k^2 - \lambda} \right| \right) \left(1 + \left| \tfrac{\lambda^-_{2k-1} - 4\pi_k^2}{4\pi_k^2 - \lambda} \right| \right) \\ &\leq \left(1 + \rho' \left| \tfrac{\lambda^-_{2k} - 4\pi_k^2}{k^2 - n^2} \right| \right) \left(1 + \rho' \left| \tfrac{\lambda^-_{2k-1} - 4\pi_k^2}{k^2 - n^2} \right| \right) \end{aligned} \tag{9.13}$$

Using that $\sum_{k \geq 1} k^{-2} \leq \pi^2/6$, one has by Cauchy-Schwarz

$$\sum_{k \neq n,n-1} \left| \frac{\lambda^-_{2k} - 4\pi_k^2}{k^2 - n^2} \right|, \quad \sum_{k \neq n,n-1} \left| \frac{\lambda^-_{2k-1} - 4\pi_k^2}{k^2 - n^2} \right| \leq \pi K \quad \text{where}$$

$$K := \big(\sum_{k \geq 1} |\lambda^-_{2k} - 4\pi_k^2|^2 + |\lambda^-_{2k-1} - 4\pi_k^2|^2 \big)^{\frac{1}{2}}.$$

Hence, uniformly for $\lambda \in \Lambda^-_{n,\varrho}$, $n_0 \leq n \leq M$,

$$|f^-_n(\lambda)| = \prod_{k \neq n,n-1} \frac{(\lambda^-_{2k} - \lambda)(\lambda^-_{2k-1} - \lambda)}{(4\pi_k^2 - \lambda)^2}$$

$$\leq \exp \left(\sum_{k \neq n,n-1} \log \big(1 + \rho' \left| \frac{\lambda^-_{2k} - 4\pi_k^2}{k^2 - n^2} \right| \big) + \sum_{k \neq n,n-1} \log \big(1 + \rho' \left| \frac{\lambda^-_{2k-1} - 4\pi_k^2}{k^2 - n^2} \right| \big) \right)$$

$$\leq \exp(2\rho'\pi K).$$

Altogether, $Q_n^-(\lambda) = O(n^2)$ uniformly for $\lambda \in \Lambda_{n,\varrho}^-$, $n_0 \leq n \leq M$ as claimed. $\square$

Proof of Theorem 2.3: By Proposition 9.1 the factor q_N^{-2} appearing in the product representation (9.1) of $\Delta_N^2 - 4$ satisfies the asymptotics

$$q_N^{-2} = 1 + O\big(\frac{1}{N^3}\big).$$

Combining Lemma 9.2, Lemma 9.3, and Lemma 9.4 one obtains, uniformly for λ in $\Lambda_{n,\rho}^-$ with $1 \leq n \leq F(M)$

$$\Delta_N^2(-2+\frac{1}{4N^2}\lambda) - 4 = \big[\Delta_-^2(\lambda) - 4 + O(\frac{M^2}{n^3 N})Q_n^-(\lambda)\big]\big(1+O(\frac{n^2}{M})\big)\big(1+O(\frac{M^2}{N})\big).$$

As $Q_n^-(\lambda) = O(n^2)$ (Lemma 9.5) and $\Delta_-^2(\lambda) - 4 = O(1)$ uniformly for $\lambda \in \Lambda_{n,\varrho}^-$ with $1 \leq n \leq F(M)$ it then follows that

$$\Delta_N^2(-2+\frac{1}{4N^2}\lambda) - 4 = \Delta_-^2(\lambda) - 4 + O(\frac{F(M)^2}{M}).$$

To determine how the signs of Δ_N and Δ_- are related note that for λ in the set

$$\{z \in \mathbb{C} \mid \quad \mathrm{dist}(z, [\lambda_{2n-1}^-, \lambda_{2n}^-]) < 2\varrho\}, \quad 1 \leq n \leq F(M),$$

one has

$$\Delta_N(-2+\frac{1}{4N^2}\lambda) = (-1)^{N-n} \sqrt[+]{\Delta_N^2(-2+\frac{1}{4N^2}\lambda)} \quad \text{and} \quad \Delta_-(\lambda) = (-1)^n \sqrt[+]{\Delta_-^2(\lambda)}.$$

Hence

$$\Delta_N(-2+\frac{1}{4N^2}\lambda) = (-1)^N \Delta_-(\lambda) + O(\frac{F(M)^2}{M}).$$

The estimates for λ in $\Lambda_{n,\varrho}^+$ with $1 \leq n \leq F(M)$ are obtained in a similar fashion. Finally, to see that these estimates are uniform on bounded sets of α, β in $C_0^2(\mathbb{T}, \mathbb{R})$ it suffices to note that ρ of (9.5) can be chosen uniformly on such sets as the periodic eigenvalues of $-\partial_x^2 + q_\pm$ are compact functions of α, β – see [22], Proposition B.11). $\square$

As $\Delta_N(\mu)$ and $\Delta_-(\lambda)$ are analytic functions one can apply Cauchy's theorem to deduce from Theorem 2.3 corresponding estimates of the derivatives $\partial_\mu^j \Delta_N$ or equivalently $\partial_\lambda^j \Delta_N(-2+\frac{\lambda}{4N^2}) = \frac{1}{(4N^2)^j}\partial_\mu^j \Delta_N\left(-2+\frac{\lambda}{4N^2}\right)$ as well as $\partial_\lambda^j \Delta_N\left(2-\frac{\lambda}{4N^2}\right) = \frac{(-1)^j}{(4N^2)^j}\partial_\mu^j \Delta_N\left(2-\frac{\lambda}{4N^2}\right)$. Let

$$\Lambda_1^{\pm,M} = [\lambda_0^+ - 1, \lambda_{2[F(M)]}^\pm + 1] + i[-1,1].$$

Corollary 9.6 *Let F satisfy* (F), *$M = [F(N)]$ with $N \geq N_0$, and $\alpha, \beta \in C_0^2(\mathbb{T}, \mathbb{R})$. Then, for any $j \geq 1$ and uniformly for λ in $\Lambda_1^{-,M}$,*

$$\frac{1}{(4N^2)^j} \partial_\mu^j \Delta_N \left(-2 + \frac{1}{4N^2} \lambda \right) = (-1)^N \partial_\lambda^j \Delta_-(\lambda) + O\left(\frac{F(M)^2}{M} \right)$$

and similarly, for λ in $\Lambda_1^{\pm,M}$

$$\frac{(-1)^j}{(4N^2)^j} \partial_\mu^j \Delta_N \left(2 - \frac{1}{4N^2} \lambda \right) = \partial_\lambda^j \Delta_+(\lambda) + O\left(\frac{F(M)^2}{M} \right).$$

These estimates hold uniformly on bounded sets of functions α, β in $C_0^2(\mathbb{T}, \mathbb{R})$.

Proof: By Cauchy's theorem, for $j \geq 1$,

$$\frac{1}{(4N^2)^j} \partial_\mu^j \Delta_N \left(-2 + \frac{1}{4N^2} \lambda \right) = \frac{1}{j!} \frac{1}{2\pi i} \int_{\partial \Lambda^{-,M}} \frac{\Delta_N(-2 + \frac{1}{4N^2} z)}{(z - \lambda)^{1+j}} dz$$

and

$$\partial_\lambda^j \Delta_-(\lambda) = \frac{1}{j!} \frac{1}{2\pi i} \int_{\partial \Lambda^{-,M}} \frac{\Delta_-(z)}{(z - \lambda)^{1+j}} dz$$

where $\partial \Lambda^{-,M}$ denotes the boundary of the rectangle $\Lambda^{-,M} \equiv \Lambda_2^{-,M}$ with counter-clockwise orientation. Hence

$$\frac{\partial_\mu^j \Delta_N \left(-2 + \lambda/4N^2 \right)}{(4N^2)^j} - (-1)^N \partial_\lambda^j \Delta_-(\lambda) = \frac{1}{j!} \frac{1}{2\pi i} \int_{\partial \Lambda_2^{-,M}} \frac{\Delta_N(-2 + \frac{1}{4N^2} z) - (-1)^N \Delta_-(z)}{(z - \lambda)^{j+1}} dz.$$

For λ in $\Lambda_1^{-,M}$, $|z - \lambda|^2 \geq 1$ and hence by Theorem 2.3, uniformly on $\Lambda_1^{-,M}$

$$\begin{aligned} \frac{\partial_\mu^j \Delta_N \left(-2 + \lambda/4N^2 \right)}{(4N^2)^j} - \partial_\lambda^j \Delta_-(\lambda) &= \frac{1}{j!} \frac{1}{2\pi i} \int_{\partial \Lambda_2^{-,M}} \frac{\Delta_N(-2 + \frac{1}{4N^2} z) - \Delta_-(z)}{(z - \lambda)^{j+1}} dz \\ &= O\left(\frac{F(M)^2}{M} \right). \end{aligned}$$

By this argument, also the uniformity statement with respect to α, β follows. $\square$

Corollary 9.6 allows to obtain asymptotics of the zeroes of $\dot{\Delta}_N(\mu) := \frac{d}{d\mu} \Delta_N(\mu)$ at the edges in terms of the zeroes of $\dot{\Delta}_\pm(\lambda) := \frac{d}{d\lambda} \Delta_\pm(\lambda)$. One sees in a straightforward way that the $N - 1$ zeroes of the polynomial $\dot{\Delta}_N(\mu)$ are all real and simple and when listed in increasing order, satisfy $\lambda_{2n-1}^N \leq \dot{\lambda}_n^N \leq \lambda_{2n}^N$ for any $0 < n < N$. Similarly one sees that the zeroes of $\dot{\lambda}_n^\pm$ are all real and simple, and when listed in increasing order, satisfy $\lambda_{2n-1}^\pm \leq \dot{\lambda}_n^\pm \leq \lambda_{2n}^\pm$ for any $n \geq 1$.

Corollary 9.7 *Let F satisfy* (F), $M = [F(N)]$, *and* $\alpha, \beta \in C_0^2(\mathbb{T}, \mathbb{R})$. *Then for any* $1 \leq n \leq F(M)$,

$$\dot{\lambda}_n^N = -2 + \frac{\dot{\lambda}_n^-}{4N^2} + O\left(\frac{n^2}{N^2}\frac{F(M)^2}{M}\right) \quad \textit{and} \quad \dot{\lambda}_{N-n}^N = 2 - \frac{\dot{\lambda}_n^+}{4N^2} + O\left(\frac{n^2}{N^2}\frac{F(M)^2}{M}\right).$$

These estimates hold uniformly on bounded sets of functions α, β *in* $C_0^2(\mathbb{T}, \mathbb{R})$.

Proof: The asymptotics of the zeroes of $\dot{\Delta}_N(\mu)$ at the two edges are obtained in a similar fashion so we concentrate on the ones at the left edge. Let Γ_n^- be the contour of the box $[\lambda_{2n-1}^- - \rho, \lambda_{2n}^- + \rho] + i[-1, 1]$, contained in $\Lambda_1^{-,M}$,

$$\Gamma_n^- = \partial([\lambda_{2n-1}^- - \rho, \lambda_{2n}^- + \rho] + i[-1, 1])$$

where ρ is chosen as in (9.5). By Theorem 2.1, for N sufficiently large, $\dot{\lambda}_n^N$ is the only zero of $\dot{\Delta}_N(\mu)$ in the box $-2 + \frac{1}{4N^2}([\lambda_{2n-1}^- - \rho, \lambda_{2n}^- + \rho] + i[-1, 1])$. In particular, $\dot{\Delta}_N(\mu)$ doesn't vanish on the contour $\Gamma_n^N = -2 + \frac{1}{4N^2}\Gamma_n^-$. By Cauchy's theorem it then follows that for any $1 \leq n \leq F(M)$, $1 = \frac{1}{2\pi i}\int_{\Gamma_n^N} \frac{\partial_\mu^2 \Delta_N}{\partial_\mu \Delta_N} d\mu$ and $\dot{\lambda}_n^N = \frac{1}{2\pi i}\int_{\Gamma_n^N} \mu \frac{\partial_\mu^2 \Delta_N}{\partial_\mu \Delta_N} d\mu$. Hence

$$\dot{\lambda}_n^N = -2 + \frac{1}{2\pi i}\int_{\Gamma_n^N} (\mu + 2)\frac{\partial_\mu^2 \Delta_N(\mu)}{\partial_\mu \Delta_N(\mu)}) d\mu$$

and with the change of variable $\mu = -2 + \frac{\lambda}{4N^2}$

$$4N^2(\dot{\lambda}_n^N + 2) = \frac{1}{2\pi i}\int_{\Gamma_n^-} \lambda \frac{\partial_\lambda^2 \Delta_N\left(-2 + \frac{\lambda}{4N^2}\right)}{\partial_\lambda \Delta_N\left(-2 + \frac{\lambda}{4N^2}\right)} d\lambda.$$

Similarly one has $\dot{\lambda}_n^- = \frac{1}{2\pi i}\int_{\Gamma_n^-} \lambda \frac{\partial_\lambda^2 \Delta_-(\lambda)}{\partial_\lambda \Delta_-(\lambda)} d\lambda$. The difference $4N^2(\dot{\lambda}_n^N + 2) - \dot{\lambda}_n^-$ thus equals

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\Gamma_n^-} \lambda \left(\frac{\partial_\lambda^2 \Delta_N\left(-2 + \frac{\lambda}{4N^2}\right)}{\partial_\lambda \Delta_N\left(-2 + \frac{\lambda}{4N^2}\right)} - \frac{(-1)^N \partial_\lambda^2 \Delta_-(\lambda)}{(-1)^N \partial_\lambda \Delta_-(\lambda)}\right) d\lambda \\
&= \frac{1}{2\pi i}\int_{\Gamma_n^-} \lambda \frac{\partial_\lambda^2 \Delta_N\left(-2 + \frac{\lambda}{4N^2}\right) - (-1)^N \partial_\lambda^2 \Delta_-(\lambda)}{\partial_\lambda \Delta_N\left(-2 + \frac{\lambda}{4N^2}\right)} d\lambda \\
&+ \frac{1}{2\pi i}\int_{\Gamma_n^-} \lambda \frac{\partial_\lambda^2 \Delta_-(\lambda) \cdot \left((-1)^N \partial_\lambda \Delta_-(\lambda) - \partial_\lambda \Delta_N\left(-2 + \frac{\lambda}{4N^2}\right)\right)}{\partial_\lambda \Delta_N\left(-2 + \frac{\lambda}{4N^2}\right) \partial_\lambda \Delta_-(\lambda)} d\lambda.
\end{aligned}$$

The two latter integrals are estimated separately. Use Corollary 9.6 and the facts that on Γ_n^-, $\lambda = O(n^2)$ and

$$\partial_\lambda^2 \Delta_-(\lambda), \qquad \frac{1}{\partial_\lambda \Delta_N\big(-2+\frac{\lambda}{4N^2}\big)}, \quad \frac{1}{\partial_\lambda \Delta_-(\lambda)} \quad = O(1)$$

to conclude that each of the two integrals is $O\left(n^2 \frac{F(M)^2}{M}\right)$, yielding

$$4N^2(\dot\lambda_n^N + 2) = \dot\lambda_n^- + O\left(n^2 \frac{F(M)^2}{M}\right).$$

The statement on the uniformity of the estimates is obtained by using that a corresponding one for the discriminants and their derivatives holds. $\square$

A Auxiliary results

In this appendix we prove auxiliary results needed to compute the asymptotics of the discriminant.

Lemma A.1 *For $N \to \infty$,*

$$\prod_{n=1}^{N-1} \big(1 - \cos \frac{n\pi}{N}\big) = 2N 2^{-N}(1 + O(N^{-1})). \tag{A.1}$$

Proof: Note that $1 - \cos(n\delta) > 0$ for $1 \leq n \leq N$, and $\delta := \pi/N$. To compute the product in (A.1) we therefore can take the logarithm, yielding,

$$\sum_{n=1}^{N} \log(1 - \cos(n\delta)) = \sum_{n=1}^{N} \log\big(\frac{1-\cos(n\delta)}{(n\delta)^2}\big) + \sum_{n=1}^{N} \log(n\delta)^2.$$

Clearly

$$\sum_{n=1}^{N} \log(n\delta)^2 = N \log \delta^2 + \log(N!)^2 = \log\big((\frac{\pi}{N})^N N!\big)^2.$$

To compute the asymptotics of $\sum_{n=1}^{N} \log\big(\frac{1-\cos(n\delta)}{(n\delta)^2}\big)$ introduce

$$f(x) = \log\big(\frac{1-\cos x}{x^2}\big) \quad 0 \leq x \leq \pi.$$

Note that for $0 \leq x \leq \pi$,

$$\frac{1-\cos x}{x^2} = \frac{1}{2} - \frac{1}{4!}x^2 + \ldots = \frac{1}{2}(1 - \frac{1}{12}x^2 + \ldots) > 0.$$

Hence $f(x)$ is a well-defined, smooth function on the interval $[0, \pi]$. Now apply the well known formula for approximating the sum $\sum_{n=1}^{N} f(n\delta)$ by an integral (cf e.g. [1])

$$\sum_{n=1}^{N} f(n\delta) = \frac{1}{\delta} \int_0^\pi f(x)dx + \frac{f(\pi) - f(0)}{2} + O(\delta) \tag{A.2}$$

where the error term $O(\delta)$ is bounded by

$$\delta \frac{1}{12} \sup_{0 \le x \le \pi} |f''(x)| \cdot \text{length}([0, \pi]).$$

Clearly

$$\frac{f(\pi) - f(0)}{2} = \frac{1}{2}\big(\log(\frac{2}{\pi^2}) - \log \frac{1}{2}\big) = \log \frac{2}{\pi}.$$

Further, $\frac{1}{\delta} \int_0^\pi f(x)dx = \frac{1}{\delta} \int_0^\pi (\log(1 - \cos x) - 2\log x)dx$ can be explicitly computed by Lemma A.2,

$$\frac{1}{\delta} \int_0^\pi \log(1 - \cos x) = -\frac{\pi}{\delta} \log 2 = \log \frac{1}{2^N}$$

and

$$-\frac{2}{\delta} \int_0^\pi \log x dx = -\frac{2}{\delta}(x \log x - x) \big|_0^\pi = 2N + \log \frac{1}{\pi^{2N}}.$$

Combining all these estimates yields

$$\sum_{n=1}^{N} \log(1 - \cos(n\delta)) = \log \big(\frac{2}{\pi} \frac{1}{2^N} \frac{1}{\pi^{2N}}\big) + 2N + \log \big(\frac{\pi^N}{N^N} N!\big)^2 + O(\frac{1}{N})$$

or

$$\prod_{n=1}^{N} \big(1 - \cos(\frac{n\pi}{N})\big) = \frac{2}{\pi} \frac{e^{2N}}{2^N N^{2N}} (N!)^2 \big(1 + O(\frac{1}{N})\big).$$

By Stirling's formula, $N! = \sqrt{2\pi N} N^N e^{-N} \big(1 + O(\frac{1}{N})\big)$ it follows

$$\prod_{n=1}^{N} \big(1 - \cos \frac{n\pi}{N}\big) = \frac{4N}{2^N} \big(1 + O(\frac{1}{N})\big)$$

and as $\big(1 - \cos \frac{n\pi}{N}\big) \big|_{n=N} = 2$ we then conclude that

$$\prod_{n=1}^{N-1} \big(1 - \cos \frac{n\pi}{N}\big) = \frac{2N}{2^N} \big(1 + O(\frac{1}{N})\big). \qquad \square$$

Lemma A.2 $\int_0^\pi \log(1 \pm \cos x)dx = -\pi \log 2.$

Proof: First note that by the change of variable of integration $x := \pi - s$,

$$\int_0^\pi \log(1+\cos x)dx = \int_0^\pi \log(1+\cos(\pi - s))ds = \int_0^\pi \log(1-\cos s)ds.$$

Hence, with $I := \int_0^\pi \log(1-\cos x)dx$, one has

$$2I = \int_0^\pi (\log(1+\cos x) + \log(1-\cos x))dx = 2\int_0^{\pi/2} \log(\sin^2 x)dx.$$

Using that $\sin^2 x = \frac{1}{2}(1-\cos 2x)$ and making the change of variable $s = 2x$, one gets $I = \frac{1}{2}\int_0^\pi \log(1-\cos s)ds - \frac{\pi}{2}\log 2$ and the claim follows. □

Lemma A.3 *For any* $1 \leq M < N$,

$$\big(\frac{\pi^2}{2N^2}\big)^M (M!)^2 \geq \prod_{n=1}^M \big(1-\cos(\frac{n\pi}{N})\big) \geq \big(\frac{\pi^2}{2N^2}\big)^M (M!)^2 \exp\big(-O(\frac{M^3}{N^2})\big). \qquad \text{(A.3)}$$

Proof: As in the proof of Lemma A.1, consider the logarithm of the product in (A.3), to obtain, with $\delta := \pi/N$,

$$\sum_{n=1}^M \log(1-\cos(n\delta)) = \sum_{n=1}^M \big(\log\frac{(n\delta)^2}{2} + \log(1+\frac{2b_n}{(n\delta)^2})\big)$$

where $b_n = 1 - \cos n\delta - \frac{(n\delta)^2}{2}$. Clearly

$$\sum_{n=1}^M \log\frac{(n\delta)^2}{2} = \log\big(\frac{(M!)^2}{2^M}(\frac{\pi}{N})^{2M}\big) = \log\big((M!)^2(\frac{\pi^2}{2N^2})^M\big). \qquad \text{(A.4)}$$

Further not that $\frac{2b_n}{(n\delta)^2} < 0$ and

$$\Big|\frac{2b_n}{(n\delta)^2}\Big| = \Big| -\frac{2}{4!}(n\delta)^2 + \frac{2}{6!}(n\delta)^4 - \Big| \leq \frac{1}{12}(n\delta)^2 \leq \frac{M^2}{N^2}.$$

As for $-1 < x < 0$,

$$0 > \log(1+x) = -\big(|x| + \frac{|x|^2}{2} + \frac{|x|^3}{3} + \dots\big) \geq -|x|\frac{1}{1-|x|}$$

it then follows that

$$0 < -\log\big(1+\frac{2b_n}{(n\delta)^2}\big) < \frac{M^2}{N^2}\big(1-\frac{M^2}{N^2}\big)^{-1} = O\big(\frac{M^2}{N^2}\big).$$

Summing up these estimates yields

$$0 < -\sum_{n=1}^{M} \log\big(1 + \frac{2b_n}{(n\delta)^2}\big) \leq M \cdot O(\frac{M^2}{N^2}) = O(\frac{M^3}{N^2}).$$

Combined with the estimate (A.4) one gets the claimed estimate. □

Lemma A.4 *For any* $1 \leq M < N$

$$2^M \geq \prod_{n=1}^{M} \big(1 + \cos\frac{n\pi}{N}\big) \geq 2^M \exp\big(-O(\frac{M^3}{N^2})\big).$$

Proof: Note that $1 + \cos\frac{n\pi}{N} = 2 - \frac{1}{2}\big(\frac{n\pi}{N}\big)^2 + \ldots = 2\big(1 - (\frac{n\pi}{2N})^2 + \ldots\big)$. Thus

$$\prod_{n=1}^{M} \big(1 + \cos\frac{n\pi}{N}\big) = 2^M \prod_{n=1}^{M} \big(1 - (\frac{n\pi}{2N})^2 + \ldots\big) \leq 2^M$$

and

$$\prod_{n=1}^{M} \big(1 + \cos\frac{n\pi}{N}\big) = 2^M \exp\big(\sum_{n=1}^{M} \log\big(1 - (\frac{n\pi}{2N})^2 + \ldots\big)\big) \geq$$

$$\geq 2^M \exp\big(-(\frac{\pi}{2N})^2 \sum_{n=1}^{M} n^2\big) \geq 2^M \exp\big(-O(\frac{M^3}{N^2})\big). \qquad \square$$

Finally we compute the spectral data for the operator $-d^2/dx^2$ when considered with periodic / antiperiodic boundary conditions on the interval $[0, T]$. The fundamental solutions of $-d^2/dx^2$ are given by $y_1(x, \lambda) = \cos\sqrt{\lambda}x$ and $y_2(x, \lambda) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}$. Thus the periodic / antiperiodic eigenvalues are

$$\lambda_0^T = 0;\ \lambda_{2n}^T = \lambda_{2n-1}^T = \big(\frac{n\pi}{T}\big)^2 \quad \forall n \geq 1$$

and a basis of eigenfunctions is given by

$$f_0 = 1; \quad f_{2n}(x) = \cos\big(\frac{n\pi}{T}x\big); \quad f_{2n-1}(x) = \sin\big(\frac{n\pi}{T}x\big).$$

The discriminant can be computed to be

$$\Delta_T(\lambda) = y_1(T, \lambda) + y_2'(T, \lambda) = 2\cos(\sqrt{\lambda}T)$$

hence

$$\Delta_T(\lambda)^2 - 4 = 4\cos^2(\sqrt{\lambda}T) - 4 = -4\sin^2(\sqrt{\lambda}T).$$

As $\sin\sqrt{\mu} = \sqrt{\mu}\prod_{n\geq 1}\frac{n^2\pi^2-\mu}{n^2\pi^2}$, it then follows that

$$\Delta_T(\lambda)^2 - 4 = -4\lambda T^2 \prod_{n\geq 1} (\frac{n^2\pi^2 - \lambda T^2}{n^2\pi^2})^2 = -4T^2\lambda \prod_{n\geq 1} (\frac{\frac{n^2\pi^2}{T^2} - \lambda}{\frac{n^2\pi^2}{T^2}})^2.$$

In view of the values of λ_n^T, it follows that

$$\Delta_T(\lambda)^2 - 4 = -4T^2\lambda \prod_{n\geq 1} \frac{(\lambda_{2n}^T - \lambda)(\lambda_{2n-1}^T - \lambda)}{(\frac{n\pi}{T})^4}.$$

Furthermore we compute the entire functions $\psi_k^T(\lambda)$, $k \geq 1$, leading to the normalized differentials $\frac{\psi_k^T(\lambda)}{\sqrt{\Delta_T^2(\lambda)-4}}d\lambda$ characterized by

$$\frac{1}{2\pi}\int_{\Gamma_n^T} \frac{\psi_k^T(\lambda)}{\sqrt[c]{\Delta_T^2(\lambda) - 4}}d\lambda = \delta_{n,k} \quad \forall n, k \geq 1$$

where, as usual, Γ_n^T is a counterclockwise contour around $\lambda_{2n}^T = \lambda_{2n-1}^T$, so that all other eigenvalues λ_k^T, $k \neq 2n, 2n-1$, are in the exterior of Γ_n^T. We claim that

$$\psi_n^T(\lambda) = c_k^T \prod_{l\neq k} \frac{\sigma_l^{T,k} - \lambda}{\left(\frac{l\pi}{T}\right)^2} \quad \text{with} \quad \sigma_l^{T,k} = \left(\frac{l\pi}{T}\right)^2 \text{ and } c_k^T = \frac{2T^2}{k\pi}.$$

Indeed, as $\sigma_l^{T,k}$ is in the ℓ'th gap interval, it follows that $\sigma_l^{T,k} = \left(\frac{l\pi}{T}\right)^2$, $\forall l \neq k$. The constant c_k^T is then determined by $1 = \frac{1}{2\pi}\int_{\Gamma_k^T} \frac{\psi_k^T(\lambda)}{\sqrt[c]{\Delta_T^2(\lambda)-4}}d\lambda$. As

$$\frac{\psi_k^T(\lambda)}{\sqrt[c]{\Delta_T^2(\lambda) - 4}} = c_k^T \frac{1}{i2T\sqrt{\lambda}} \frac{\left(\frac{k\pi}{T}\right)^2}{\lambda - \lambda_{2k}^T}$$

one gets by Cauchy's Theorem that $c_k^T = \frac{2T^2}{k\pi}$ as claimed. In the special case where $T = 1/2$ one gets

$$\Delta(\lambda)^2 - 4 = -\lambda \prod_{n\geq 1} \frac{(\lambda_{2n} - \lambda)(\lambda_{2n-1} - \lambda)}{(2n\pi)^4}$$

where $\lambda_n \equiv \lambda_n^T\Big|_{T=1/2}$ for any $n \geq 0$ and $\Delta(\lambda) \equiv \Delta_T(\lambda)\Big|_{T=1/2}$. For the entire functions $\psi_k(\lambda) := \psi_k^T(\lambda)|_{T=\frac{1}{2}}$ one gets

$$\psi_k(\lambda) = \frac{1}{2k\pi}\prod_{l\neq k} \frac{(2l\pi)^2 - \lambda}{(2l\pi)^2} \quad \text{and} \quad c_k := c_k^T|_{T=\frac{1}{2}} = \frac{1}{2k\pi}.$$

References

[1] H. Amann, J. Escher: *Analysis II*, Birkhäuser, Basel, 1999.

[2] D. Bambusi, T. Kappeler, T. Paul: *De Toda à KdV*, C. R. Acad. Sci. Paris, Ser I, 347(17-18), 1025-1030, 2009.

[3] D. Bambusi, T. Kappeler, T. Paul: *Dynamics of periodic Toda chains with a large number of particles,* J. of Differential Equations 258, 4209-4274, 2015.

[4] D. Bambusi, A. Ponno: *On metastability in FPU*, Comm. Math. Phys. 264, 539-561, 2006.

[5] G. Benettin, H. Christodoulidi, A. Ponno: *The Fermi-Pasta-Ulam problem and its underlying integrable dynamics*, J. Stat. Phys. 144, 793, 2001.

[6] L. Berchialla, L. Galgani, A. Giorgilli: *Localisation of energy in FPU chains* DCDS-A 11, 855 - 866, 2005.

[7] G. Berman, F. Izrailev: *The Fermi-Pasta-Ulam problem: 50 years of progress*, Chaos 15, 015104, 2005.

[8] J. Biello, P. Kramer, Y. Lvov, *Stages of energy transfer in the FPU model,* Dynamical systems and differential equations (Wilmington, NC, 2002), DCDS Suppl., 113 -122, 2003.

[9] A. Bloch, F. Golse, T. Paul, A. Uribe: *Dispersionless Toda and Toeplitz operators*, Duke Math. J. 117, 157-196, 2003.

[10] A. Carati, L. Galgani, A. Giorgilli: *The Fermi– Pasta–Ulam problem as a challenge for the foundations in physics,* Chaos, 15, 015105, 2005.

[11] C. Cercigniani: *Solitons. Theory and application*, Riv. Nuovo Cim. 7, 429-469, 1977.

[12] W. Ferguson, H. Flaschka, D. McLaughlin: *Nonlinear Toda modes for the Toda chain,* J. Comput. Phys. 45, 157-209, 1982.

[13] E. Fermi, J. Pasta, S. Ulam: *Los Alamos report LA – 1940*, in E. Fermi, *Collected Papers*, vol 2, 977-988, 1955.

[14] H. Flaschka: *The Toda lattice I. Existence of integrals*, Phys. Rev. B 9, 1924-1925, 1974.

[15] E. Fucito, F. Marchesoni, E. Marinari, G. Parisi, L. Peliti, S. Ruffo, A. Vulpiani: *Approach to equilibrium in a chain of nonlinear oscillators*, J. de Physique 43, 707-713, 1982.

[16] G. Gallavotti (Ed.): *Fermi–Pasta–Ulam problem: a status report*, Lect. Notes Phys., vol 728, Springer, 2007.

[17] D. Gieseker: *The Toda hierarchy and the KdV hierarchy*, Comm. Math. Phys. 181, 587-603, 1996.

[18] D. Gieseker: *Toda and KdV*, J. of Differential Geometry 64, 171-246, 2003.

[19] A. Henrici, T. Kappeler: *Global action-angle variables for the periodic Toda lattice*, IMRN 2008; Vol 2008: article ID run 031, 52 pages, doi:10.1093/imrn/rnn031.

[20] A. Henrici, T. Kappeler: *Results on normal forms for FPU chains*, Comm. Math. Phys. 278, 14-177, 2008.

[21] A. Henrici, T. Kappeler: *Resonant normal forms for even periodic FPU chains*, J. European Math. Soc. 11, 1025-1056, 2009.

[22] T. Kappeler, J. Pöschel: *KdV & KAM*, Springer, Heidelberg, 2003.

[23] R. Livi, M. Pettini, S. Ruffo, M. Sparpaglione, A. Vulpiani: *Relaxation to different stationary states in the Fermi–Pasta–Ulam model*, Phys. Rev. A 28, 3544-3552, 1983.

[24] R. Livi, M. Pettini, S. Ruffo, A. Vulpiani: *Further results on the equipartition threshold in large nonlinear Hamiltonian systems*, Phys. Rev. A 31, 2741-2742, 1985.

[25] C. Morosi, L. Pizzocchero: *On the continuous limit of integrable lattices. I. The Kac-Moerbeke system and KdV theory*, Commun. Math. Phys. 180 (2), 505528, 1996.

[26] T. Paul, A. Uribe: *A construction of quasimodes using coherent states*, Ann. H. Poincaré Phys. Théo. 59 (4), 357–382, 1993.

[27] M. Pettini, M. Landolfi: *Relaxation properties and ergodicity breaking in nonlinear Hamiltonian dynamics*, Phys. Rev. A 41, 768-783, 1990.

[28] J. Pöschel, E. Trubowitz: *Inverse spectral theory*, Academic Press, Boston, 1987.

[29] G. Schneider, C. Wayne: *Counter-propagating waves on fluid surfaces and the continuum limit of the Fermi-Pasta-Ulam model*, in International Conference on Differential Equations, Vol. 1.2 (Berlin 1999), 390-404, World Sci. Publ. 2000.

[30] M. Toda: *Theory of nonlinear lattices*, 2nd edition, Springer Series on Solid-State Sciences 20, Springer, 1989.

[31] N. Zabusky, M. Kruskal: *Interaction of solitons in a collisonsless plasma and the recurrence of initial states*, Phys. Rev. Lett. 15, 240 - 245, 1965.